AN INTERESTING PROBLEM RELATED TO PROGRESSION IN ALGEBRA

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ABSTRACT
Given the product value of two non-zero distinct integers x, y and the value of the ratio $\frac{\text{lcm}}{\text{gcd}}$ of x, y, a process of obtaining the values of x and y is illustrated.

Keywords: Arithmetic mean, Harmonic Mean
MSC classification number: 11D99

INTRODUCTION
Every mathematics student is familiar with the concept of Arithmetic mean, Geometric mean and Harmonic mean between any two Natural numbers. For a variety of problems on Arithmetic mean, Geometric mean and Harmonic mean, one may refer (Bernald and Child, 2006; Hall and Knight, 1998). In page 91 of the journal “The Mathematics Teacher” published by the association of the Mathematics teacher of India, there is a problem involving Arithmetic mean and Harmonic mean. In this short communication an attempt has been made to generalize the above problem.

MATERIALS AND METHOD
Method of Analysis
Let x, y be any two non-zero distinct positive integers whose product is denoted by P.
Let $p= \text{gcd} (x,y), q= \text{lcm} (x,y)$. Note that $q>p$ and $xy=pq=P$. Let A be an arithmetic mean of p,q and H be a Harmonic mean of p,q. Denote the value of the ratio $\frac{A}{H}$ by $N_1$, a rational number.

Given the values of P and $N_1$, we illustrate below a process of obtaining the values of x and y. Consider

$$\frac{A}{H} = N_1 \Rightarrow (q+p)^2 = 4PN_1$$

$$q + p = 2\sqrt{PN_1} \quad (1)$$

We have the identity

$$(q-p)^2 = (q+p)^2 - 4pq$$

$$\Rightarrow (q-p)^2 = 4PN_1 - 4P$$

$$q - p = 2\sqrt{P(N_1-1)} \quad (2)$$

Solving (1) and (2), we get

$$q = \sqrt{PN_1 + P(N_1-1)}$$

$$p = \sqrt{PN_1 - P(N_1-1)}$$

It is noteworthy that the products $PN_1$ and $P(N_1-1)$ should be perfect squares. Since p is the gcd (x,y), we write

$x = pX, y = pY$, where gcd (X,Y)=1

Now, $P = xy = p^2XY$
\[ XY = \frac{p}{p^2} \]  

(4)

Which should be an integer. Then, it is possible to choose X, Y such that gcd (X,Y)=1. Substituting the values of p, X, Y in (3), the values of x, y are obtained.

A few examples are given below.

<table>
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<tr>
<th>S.no</th>
<th>P</th>
<th>(N_1)</th>
<th>(PN_1)</th>
<th>(P(N_1-1))</th>
<th>p</th>
<th>XY</th>
<th>X</th>
<th>Y</th>
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</table>

**CONCLUSION**

To sum up, the following three conditions have to be satisfied

1) \( PN_1 \) is a perfect square
2) \( P(N_1 - 1) \) is also a perfect square
3) \( P \equiv 0 (\text{mod} \ p^2) \)

Also from the above table, it is observed that when P is a perfect square the ratio \((\text{Geometric mean of } p, q)^2 / (\text{Harmonic mean of } p, q)\) is an integer representing hypotenuse of a Pythagorean triangle.

**REFERENCES**


The Mathematics Teacher, the association of the Mathematics Teacher of India, Vol 49, issue 1 & 2, Pg 91, Problem no 7, 2013.