International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Online International Journal Available at http://www.cibtech.org/jpms.htm
2013 Vol. 3 (1) January-March, pp.51-55/Natarajan
Research Article

CAUCHY MULTIPLICATION OF (M, λ_n) SUMMABLE SERIES IN ULTRAMETRIC FIELDS

***P. N. Natarajan**
**Old No. 2/3, New No. 3/3, Second Main Road, R.A. Puram*
Chennai 600 028, India
**Author for Correspondence*

ABSTRACT
In this paper, K denotes a complete, non-trivially valued, ultrametric field. Infinite matrices, sequences and series have entries in K. The main purpose of this paper is to prove a few theorems on the Cauchy multiplication of (M, λ_n) summable series in K.
2000 Mathematics Subject Classification: 40, 46.

Key Words: *Ultrametric Field, (M, λ_n) Method, Cauchy Multiplication*

INTRODUCTION AND PRELIMINARIES

Throughout the present paper, K denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. Infinite matrices, sequences and series have entries in K. In order to make the paper self-contained, we recall the following. Given an infinite matrix $A \equiv (a_{nk})$, $a_{nk} \in K$, n, k = 0, 1, 2, ... and a sequence $x = \{x_k\}$, $x_k \in K$, k = 0, 1, 2, ..., by the A-transform of $x = \{x_k\}$, we mean the sequence $A(x) = \{(Ax)_n\}$,

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0,1,2,...,$$

Where we assume that the series on the right converge. If $\lim_{n\to\infty} (Ax)_n = \ell$, we say that $x = \{x_k\}$ is A-summable or summable A to ℓ. If $\lim_{n\to\infty} (Ax)_n = \ell$ whenever $\lim_{k\to\infty} x_k = \ell$, we say that A is regular. The following result, which gives a set of necessary and sufficient conditions for A to be regular in terms of the entries of the matrix, is well-known.

Theorem 1.1 (Monna (1963)) $A \equiv (a_{nk})$ is regular if and only if

(i) $\sup_{n,k} |a_{nk}| < \infty$;

(ii) $\lim_{n\to\infty} a_{nk} = 0, \quad k = 0,1,2,...;$

and

(iii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1.$

An infinite series $\sum_{k=0}^{\infty} x_k$ is said to be A-summable to ℓ if $\{s_n\}$ is A-summable to ℓ where $s_n = \sum_{k=0}^{n} x_k$, n = 0, 1, 2,....

The (M, λ_n) method in K was introduced earlier by Natarajan (2003) and some of its properties were studied in (Natarajan (2003, 2012a, 2012b)).

Definition 1.2 Let $\{\lambda_n\}$ be a sequence in K such that $\lim_{n\to\infty} \lambda_n = 0$. The (M, λ_n) method is defined by the infinite matrix (a_{nk}), where

$$a_{nk} = \begin{cases} \lambda_{n-k}, & k \le n; \\ 0, & k > n. \end{cases}$$

Remark 1.3 In this context, we note that the (M, λ_n) method reduces to the Y-method of Srinivasan (1965), when K = Q_p, the p-adic field for a prime p, $\lambda_0 = \lambda_1 = \frac{1}{2}$ and $\lambda_n = 0$, $n \ge 2$.

Theorem 1.4 (see Natarajan (2012b), Theorem 2.1). The (M, λ_n) method is regular if and only if

$$\sum_{n=0}^{\infty} \lambda_n = 1.$$

RESULTS

The following result is very useful in the sequel (see Natarajan (1978), Theorem 1).

Theorem 2.1 If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} c_n = 0$. Further, if $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ converge with sums A, B respectively, then $\sum_{n=0}^{\infty} c_n$ converges too with sum AB, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, n = 0, 1, 2, ...

In this paper, we prove a few theorems on the Cauchy multiplication of (M, λ_n) summable series in K.

Theorem 2.2 If $a_k = o(1)$, $k \to \infty$, i.e., $\lim_{k\to\infty} a_k = 0$ and $\{b_k\}$ is (M, λ_n) summable to B, then $\{c_k\}$ is (M, λ_n) summable to AB, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, n = 0, 1, 2, ... and $\sum_{k=0}^{\infty} a_k = A$.

Proof. Let

$t_n = \lambda_0 b_n + \lambda_1 b_{n-1} + \cdots + \lambda_n b_0$, n = 0, 1, 2, ...

By hypothesis, $\lim_{n\to\infty} t_n = B$. Let

$u_n = \lambda_0 c_n + \lambda_1 c_{n-1} + \cdots + \lambda_n c_0$, n = 0, 1, 2, ...

Then

$$\begin{aligned} u_n &= \lambda_0 (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &\quad + \lambda_1 (a_0 b_{n-1} + a_1 b_{n-2} + \cdots + a_{n-1} b_0) + \cdots + \lambda_n (a_0 b_0) \\ &= a_0 (\lambda_0 b_n + \lambda_1 b_{n-1} + \cdots + \lambda_n b_0) \\ &\quad + a_1 (\lambda_0 b_{n-1} + \lambda_1 b_{n-2} + \cdots + \lambda_{n-1} b_0) + \cdots + a_n (\lambda_0 b_0) \\ &= a_0 t_n + a_1 t_{n-1} + \cdots + a_n t_0 \\ &= a_0 (t_n - B) + a_1 (t_{n-1} - B) + \cdots + a_n (t_0 - B) \\ &\quad + B(a_0 + a_1 + \cdots + a_n). \end{aligned}$$

Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (t_n - B) = 0$, in view of Theorem 2.1,

$$\lim_{n\to\infty} [a_0(t_n - B) + a_1(t_{n-1} - B) + \ldots + a_n(t_0 - B)] = 0$$

so that

$$\lim_{n\to\infty} u_n = B\left(\sum_{n=0}^{\infty} a_n\right) = AB,$$

i.e., $\{c_k\}$ is (M, λ_n) summable to AB, completing the proof. □

It is easy to prove the following theorem on similar lines.

Theorem 2.3 If $\sum_{k=0}^{\infty} a_k$ converges to A and $\sum_{k=0}^{\infty} b_k$ is (M, λ_n) summable to B, then $\sum_{k=0}^{\infty} c_k$ is (M, λ_n) summable to AB, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, n = 0, 1, 2, ...

Following Mears (1935) and Natarajan (1997), we prove the following result.

Theorem 2.4 If $\sum_{k=0}^{\infty} a_k$ is (M, λ_n) summable to A, $\sum_{k=0}^{\infty} b_k$ is (M, μ_n) summable to B, then $\sum_{k=0}^{\infty} c_k$ is (M, γ_n) summable to AB, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, $\gamma_n = \sum_{k=0}^{n} \lambda_k \mu_{n-k}$, n = 0, 1, 2, ...

Proof. First we note that $\lim_{n\to\infty} \gamma_n = 0$ using Theorem 2.1, since $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \mu_n = 0$ so that the method (M, γ_n) is defined.

Let $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, $C_n = \sum_{k=0}^{n} c_k$, n = 0, 1, 2, ... Let

$\alpha_n = \sum_{k=0}^{n} \lambda_k A_{n-k}$, $\beta_n = \sum_{k=0}^{n} \mu_k B_{n-k}$, $\delta_n = \sum_{k=0}^{n} \gamma_k C_{n-k}$, n = 0, 1, 2, ... We now do some computation to show that

$$\delta_n = \sum_{k=0}^{n} \alpha_k \beta_{n-k} - \sum_{k=0}^{n-1} \alpha_k \beta_{n-k-1}.$$

We first note that

$$C_n = a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0,$$

so that

$$\begin{aligned}\delta_n &= \gamma_0 C_n + \gamma_1 C_{n-1} + \cdots + \gamma_n C_0\\ &= \gamma_0 (a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0)\\ &\quad + \gamma_1 (a_0 B_{n-1} + a_1 B_{n-2} + \cdots + a_{n-1} B_0) + \cdots + \gamma_n (a_0 B_0)\\ &= a_0 (\gamma_0 B_n + \gamma_1 B_{n-1} + \cdots + \gamma_n B_0)\\ &\quad + a_1 (\gamma_0 B_{n-1} + \gamma_1 B_{n-2} + \cdots + \gamma_{n-1} B_0) + \cdots + a_n (\gamma_0 B_0).\end{aligned}$$

One can prove that

$$\gamma_0 B_n + \gamma_1 B_{n-1} + \cdots + \gamma_n B_0 = \lambda_0 \beta_n + \lambda_1 \beta_{n-1} + \cdots + \lambda_n \beta_0,$$

n = 0, 1, 2, ... so that

$$\begin{aligned}\delta_n &= a_0 [\lambda_0 \beta_n + \lambda_1 \beta_{n-1} + \cdots + \lambda_n \beta_0]\\ &\quad + a_1 [\lambda_0 \beta_{n-1} + \lambda_1 \beta_{n-2} + \cdots + \lambda_{n-1} \beta_0] + \cdots + a_n [\lambda_0 \beta_0]\\ &= A_0 [\lambda_0 \beta_n + \lambda_1 \beta_{n-1} + \lambda_2 \beta_{n-2} + \lambda_3 \beta_{n-3} + \cdots + \lambda_n \beta_0]\\ &\quad + (A_1 - A_0) [\lambda_0 \beta_{n-1} + \lambda_1 \beta_{n-2} + \lambda_2 \beta_{n-3} + \cdots + \lambda_{n-1} \beta_0]\\ &\quad + (A_2 - A_1) [\lambda_0 \beta_{n-2} + \lambda_1 \beta_{n-3} + \cdots + \lambda_{n-2} \beta_0]\\ &\quad + (A_3 - A_2) [\lambda_0 \beta_{n-3} + \lambda_1 \beta_{n-4} + \cdots + \lambda_{n-3} \beta_0]\\ &\quad + \cdots + (A_n - A_{n-1}) [\lambda_0 \beta_0]\\ &= \beta_n [\lambda_0 A_0] + \beta_{n-1} [\lambda_1 A_0 + \lambda_0 (A_1 - A_0)]\\ &\quad + \beta_{n-2} [\lambda_2 A_0 + \lambda_1 (A_1 - A_0) + \lambda_0 (A_2 - A_1)]\\ &\quad + \beta_{n-3} [\lambda_3 A_0 + \lambda_2 (A_1 - A_0) + \lambda_1 (A_2 - A_1) + \lambda_0 (A_3 - A_2)]\\ &\quad + \cdots\\ &\quad + \beta_0 [\lambda_n A_0 + \lambda_{n-1} (A_1 - A_0) + \lambda_{n-2} (A_2 - A_1)\\ &\quad + \cdots + \lambda_0 (A_n - A_{n-1})]\end{aligned}$$

$$= \beta_n [\lambda_0 A_0] + \beta_{n-1} [(\lambda_0 A_1 + \lambda_1 A_0) - \lambda_0 A_0]$$
$$+ \beta_{n-2} [(\lambda_0 A_2 + \lambda_1 A_1 + \lambda_2 A_0) - (\lambda_0 A_1 + \lambda_1 A_0)]$$
$$+ \cdots$$
$$+ \beta_0 [(\lambda_0 A_n + \lambda_1 A_{n-1} + \cdots + \lambda_n A_0)$$
$$- (\lambda_0 A_{n-1} + \lambda_1 A_{n-2} + \cdots + \lambda_{n-1} A_0)]$$
$$= \beta_n \alpha_0 + \beta_{n-1} [\alpha_1 - \alpha_0] + \beta_{n-2} [\alpha_2 - \alpha_1] + \cdots + \beta_0 [\alpha_n - \alpha_{n-1}]$$
$$= (\alpha_0 \beta_n + \alpha_1 \beta_{n-1} + \cdots + \alpha_n \beta_0)$$
$$- (\alpha_0 \beta_{n-1} + \alpha_1 \beta_{n-2} + \cdots + \alpha_{n-1} \beta_0)$$
$$= \sum_{k=0}^{n} \alpha_k \beta_{n-k} - \sum_{k=0}^{n-1} \alpha_k \beta_{n-k-1},$$

proving our claim. Thus, for n = 0, 1, 2, ...,

$$\delta_n = \sum_{k=0}^{n} (\alpha_k - A)\beta_{n-k} - \sum_{k=0}^{n-1} (\alpha_k - A)\beta_{n-k-1} + A\left[\sum_{k=0}^{n} \beta_{n-k} - \sum_{k=0}^{n-1} \beta_{n-k-1}\right]$$
$$= \sum_{k=0}^{n} (\alpha_k - A)\beta_{n-k} - \sum_{k=0}^{n-1} (\alpha_k - A)\beta_{n-k-1} + A\beta_n$$
$$= \sum_{k=0}^{n} (\alpha_k - A)(\beta_{n-k} - B) - \sum_{k=0}^{n-1} (\alpha_k - A)(\beta_{n-k-1} - B) + B\left[\sum_{k=0}^{n} (\alpha_k - A) - \sum_{k=0}^{n-1} (\alpha_k - A)\right] + A\beta_n$$
$$= \sum_{k=0}^{n} (\alpha_k - A)(\beta_{n-k} - B) - \sum_{k=0}^{n-1} (\alpha_k - A)(\beta_{n-k-1} - B) + A\beta_n + B[\alpha_n - A].$$

Using Theorem 2.1, we have,

$$\lim_{n\to\infty} \sum_{k=0}^{n} (\alpha_k - A)(\beta_{n-k} - B) = 0$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} (\alpha_k - A)(\beta_{n-k-1} - B) = 0,$$

Since $\lim_{n\to\infty} \alpha_n = A$ and $\lim_{n\to\infty} \beta_n = B$. Consequently,

$$\lim_{n\to\infty} \delta_n = AB,$$

i.e., $\sum_{k=0}^{\infty} c_k$ is (M, γ_n) summable to AB, completing the proof of the theorem. □

Remark 2.5 In particular, if (M, λ_n), (M, μ_n) are regular, then $\sum_{n=0}^{\infty} \lambda_n = \sum_{n=0}^{\infty} \mu_n = 1,$ in view of Theorem 1.4. Thus

$$\sum_{n=0}^{\infty} \gamma_n = \left(\sum_{n=0}^{\infty} \lambda_n\right)\left(\sum_{n=0}^{\infty} \mu_n\right) = 1,$$

in view of Theorem 2.1. Using Theorem 1.4 again, it follows that (M, γ_n) is regular too.

REFERENCES

Bachman G (1964). Introduction to p-adic numbers and valuation theory (Academic Press).
Mears FM (1935). Some multiplication theorems for the Nörlund means. *Bulletin of the American Mathematical Society* **41** 875-880.

Monna AF (1963). Sur le théorème de Banach-Steinhaus. *Indagationes Mathématicae* **25** 121-131.

Natarajan PN (1978). Multiplication of series with terms in a non-archimedean field. *Simon Stevin* **52** 157-160.

Natarajan PN (1997). A multiplication theorem for Nörlund means in non-archimedean fields. *Journal of Analysis* **5** 59-63.

Natarajan PN (2003). Product theorems for certain summability methods in non-archimedean fields. *Annales Mathematiques Blaise Pascal* **10**(2) 261-267.

Natarajan PN (2012a). A product theorem for the Euler and M methods of summability in ultrametric fields. *Advancement and Development in Mathematical Sciences* **2**(1and 2) 31-39.

Natarajan PN (2012b). Some properties of the (M, λ_n) method of summability in ultrametric fields. *International Journal of Physics and Mathematical Sciences* **2**(2) 169-176.

Srinivasan VK (1965). On certain summation processes in the p-adic field. *Indagationes Mathematicae* **27** 368-374.