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Research Article

A NEW THEOREM DEVELOPED TO DEFINE THE PROPERTY OF PAIR CONJUGATE DIAMETERS OF AN ELLIPSE

***Kalaimaran Ara**
Construction & Civil Maintenance Dept., Central Food Technological Research Institute, Mysore-20, Karnataka, India
**Author for Correspondence*

ABSTRACT
Ellipseis one of the conic sections. It is an elongated circle. It is the locus of a point that moves in such a way that the ratio of its distance from a fixed point (called Focus) to its distance from a fixed line (called Directrix) equals to constant 'e' which is less than or equal to unity. According to Keplar's law of Planetary Motion, the ellipse is very important in geometry and the field of Astronomy, since every planet is orbiting its star in an elliptical path and its star is as one of the foci. The author has derived equations with necessary illustrations to arrive mathematical theorems for the properties of conjugate diameters of an ellipse.

Key Words: *Ellipse, Conic Sections, Diameter, Conjugate Diameter, Eccentricity of Ellipse, Foci, Semi-Major Axis, Semi-Minor Axis.*

INTRODUCTION

An *ellipse* (Weisstein Eric, 2003) is the set of all points in a plane such that the sum of the distances from two fixed points called *foci* (Weisstein Eric, 2003) is a given constant. Things that are in the shape of an ellipse are said to be elliptical. In the 17th century, a mathematician Mr. Johannes Kepler discovered that the orbits along which the planets travel around the Sun are ellipses with the Sun at one of the foci, in his First law of planetary motion. Later, Isaac Newton explained that this as a corollary of his law of universal gravitation. One of the physical properties of ellipse is that sound or light rays emanating from one focus will reflect back to the other focus. This property can be used, for instance, in medicine. A point inside the ellipse which is the midpoint of the *line* (Weisstein Eric, 2003) linking the two foci is called centre. The longest and shortest diameters of an ellipse is called *Major axis* (Weisstein Eric, 2003) and *Minor axis* (Weisstein Eric, 2003) respectively. The two points that define the ellipse is called foci. The *eccentricity* (Weisstein Eric, 2003) of an ellipse, usually denoted by ε or e, is the ratio of the distance between the two foci to the length of the major axis. A line segment linking any two points on an ellipse is called *chord* (Weisstein Eric, 2003). A straight line passing an ellipse and touching it at just one point is called *tangent* (Weisstein Eric, 2003).A straight line that passing through the centre of ellipse is called *diameter* (Weisstein Eric, 2003) A straight line that passing through the centre of the parallel lines to the diameter of the ellipse is called *conjugate diameter* (Clapham et al., 2009).The author has derived necessary equations for geometrical properties of conjugate diameter drawn to diameter of an ellipse and developed a new geometrical theorems for it.

New Theorem on Ellipse With Respect to Its Pair Conjugate Diameters

In an ellipse, if a *parallelogram* (Weisstein Eric, 2003) PSQR drawn by connecting all the *extremities* (Bali, 2005) of pair conjugate diameters drawn anywhere on an ellipse is always an arbitrary constant and its value is equal to the area of the *rhombus* (Weisstein Eric, 2003) which is formed by connecting the extremities ACBD of the ellipse.

Derivation of Equations and Proof Of The Theorem

Referring figure 1,

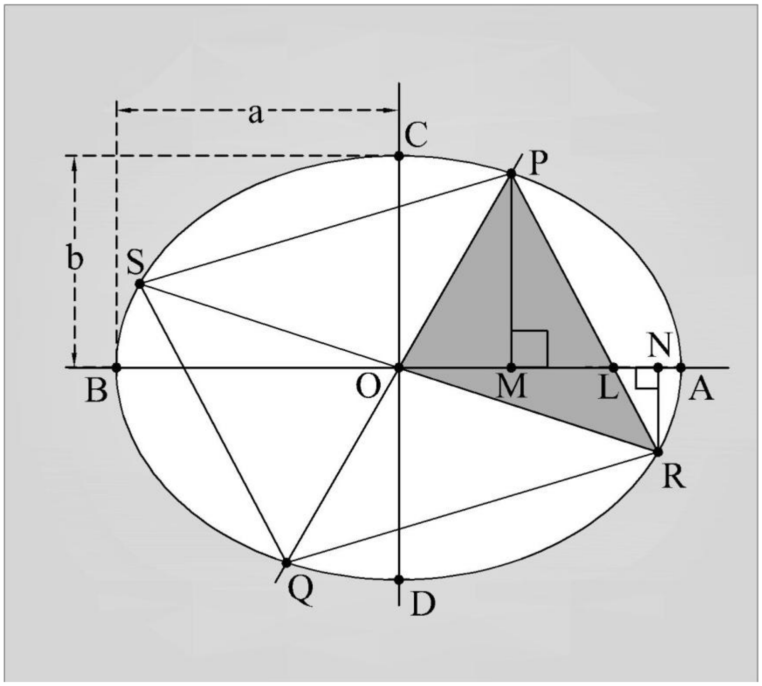

Figure 1: An Ellipse with Diameter PQ and Conjugate diameter RS to PQ. The shaded area of triangle OPR

LetABCD are the extremities of the ellipse.Line PQ and RS are the conjugate diameter anywhere in ellipse and passing through the geometric centre 'O' of the ellipse. The Parametric co-ordinates (Bali, 2005) of the points 'P' & 'Q' of the ellipse are (acosθ°, bsinθ°) and (–acosθ°, –bsinθ°) respectively. Line 'RS' is the conjugate diameter to 'PQ'. Therefore, *the parametric co-ordinates of the extremities of conjugate diameter* (Bali, 2005) 'R' & 'S' are (asinθ°, –bcosθ°) and (–asinθ°, bcosθ°) respectively.

Let, point 'M' is the projection of point 'P' on major axis, 'N' is the projection of 'R' on major axis. Therefore, PM ⊥ OA, RN ⊥ OA.

Let, coordinates of P $= (x_1, y_1)$, coordinates of R $= (x_2, y_2)$, coordinates of S $= (x_3, y_3)$.

Therefore, $(x_1,\ y_1) = (\mathrm{a}\cos\theta°, \mathrm{b}\sin\theta°), (x_2,\ y_2) = (\mathrm{a}\sin\theta°, -\mathrm{b}\cos\theta°),\ (x_3,\ y_3) = (-\mathrm{a}\sin\theta°, \mathrm{b}\sin\theta°)$

Therefore, $x_1 = \mathrm{a}\cos\theta° \text{ ------------------------------ } [1.1]$

Therefore, $y_1 = \mathrm{b}\sin\theta° \text{ ------------------------------ } [1.2]$

Therefore, $x_2 = \mathrm{a}\sin\theta° \text{ ------------------------------ } [1.3]$

Therefore, $y_2 = -\mathrm{b}\cos\theta° \text{ ------------------------------ } [1.4]$

We know that the Equation of straight linepassing through any two points $(x_1, y_1)\&(x_2, y_2)$ is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) \text{ ------------------------------ } [1.5]$$

Equation of line 'PR' passing through $P(\mathrm{a}\cos\theta°, \mathrm{b}\sin\theta°)$ and $R(\mathrm{a}\sin\theta°, -\mathrm{b}\cos\theta°)$

$$\Longrightarrow\ y - \mathrm{b}sin\theta° = \left(\frac{-\mathrm{b}cos\theta° - \mathrm{b}sin\theta°}{\mathrm{a}sin\theta° - \mathrm{a}cos\theta°}\right)(x - \mathrm{a}cos\theta°) \text{ ------------------ } [1.6]$$

Referring the figure 1, the line PR is intersecting the major axis at point L(y = 0). Therefore, substituting y = 0 in eqn.[1.6], we get

$$\Longrightarrow\ 0 - b\sin\theta^\circ = \left(\frac{b\sin\theta^\circ + b\cos\theta^\circ}{a\cos\theta^\circ - a\sin\theta^\circ}\right)(0 - a\cos\theta^\circ)$$

$$\Longrightarrow\ (x - a\cos\theta^\circ) = -b\sin\theta^\circ \times \left(\frac{a\cos\theta^\circ - a\sin\theta^\circ}{b\sin\theta^\circ + b\cos\theta^\circ}\right)$$

$$\Longrightarrow\ (x - a\cos\theta^\circ) = -a\sin\theta^\circ \times \left(\frac{\cos\theta^\circ - \sin\theta^\circ}{\sin\theta^\circ + \cos\theta^\circ}\right)$$

$$\Longrightarrow\ (x - a\cos\theta^\circ) = a\sin\theta^\circ \times \left(\frac{\sin\theta^\circ - \cos\theta^\circ}{\sin\theta^\circ + \cos\theta^\circ}\right)$$

Therefore, $x = \left\{a\sin\theta^\circ \times \left(\frac{\sin\theta^\circ - \cos\theta^\circ}{\sin\theta^\circ + \cos\theta^\circ}\right)\right\} + a\cos\theta^\circ$

Therefore, $x = \left\{\frac{a\sin\theta^\circ(\sin\theta^\circ - \cos\theta^\circ) + a\cos\theta^\circ(\sin\theta^\circ + \cos\theta^\circ)}{\sin\theta^\circ + \cos\theta^\circ}\right\}$

Therefore, $x = \frac{a\sin^2\theta^\circ - a\sin\theta^\circ\cos\theta^\circ + a\sin\theta^\circ\cos\theta^\circ + a\cos^2\theta^\circ}{\sin\theta^\circ + \cos\theta^\circ}$

Therefore, $x = \frac{a\sin^2\theta^\circ + a\cos^2\theta^\circ}{\sin\theta^\circ + \cos\theta^\circ}$

Therefore, $x = a\left(\frac{\sin^2\theta^\circ + \cos^2\theta^\circ}{\sin\theta^\circ + \cos\theta^\circ}\right)$

Therefore, $OL = \frac{a}{\sin\theta^\circ + \cos\theta^\circ}$ ------------------------ [1.7]

Referring figure 1, Let Area of triangle $OLP = A_1$

Therefore, $A_1 = \frac{1}{2} \times OL \times PM$

Substituting, eqn.[1.7] & [1.2] in above equation, we get

Therefore, $A_1 = \frac{1}{2} \times \left(\frac{a}{\sin\theta^\circ + \cos\theta^\circ}\right) \times b\sin\theta^\circ$

Therefore, $A_1 = \frac{ab\sin\theta^\circ}{2(\sin\theta^\circ + \cos\theta^\circ)}$ ------------------------[1.8]

Referring figure 1, Let Area of triangle $OLR = A_2$

Therefore, $A_2 = \frac{1}{2} \times OL \times RN$

Substituting, eqn.[1.7] & [1.4] in above equation, we get

Therefore, $A_2 = \frac{1}{2} \times \left(\frac{a}{\sin\theta^\circ + \cos\theta^\circ}\right) \times b\cos\theta^\circ$

Therefore, $A_2 = \frac{ab\cos\theta^\circ}{2(\sin\theta^\circ + \cos\theta^\circ)}$ ------------------------[1.9]

The total area of ΔOPR = Area of ΔOLP + Area of ΔOLR

Toatal area of $\Delta OPR = A_1 + A_2$

Substituting, [1.8]&[1.9]in above, we get

$$A = \frac{ab\sin\theta^\circ}{2(\sin\theta^\circ + \cos\theta^\circ)} + \frac{ab\cos\theta^\circ}{2(\sin\theta^\circ + \cos\theta^\circ)}$$

Therefore, $A = \frac{ab}{2}\left(\frac{\sin\theta^\circ + \cos\theta^\circ}{\sin\theta^\circ + \cos\theta^\circ}\right)$

Therefore, $A = \frac{ab}{2}$

Therefore, the area of $\Delta OPR = \frac{ab}{2}$ -------------------- [1.10]

In figure 2, 'J' is the projection of point 'P' on minor axis and point 'K' is the projection of point 'S' on minor axis.Therefore, PJ ⊥ OC and SK ⊥ OC.

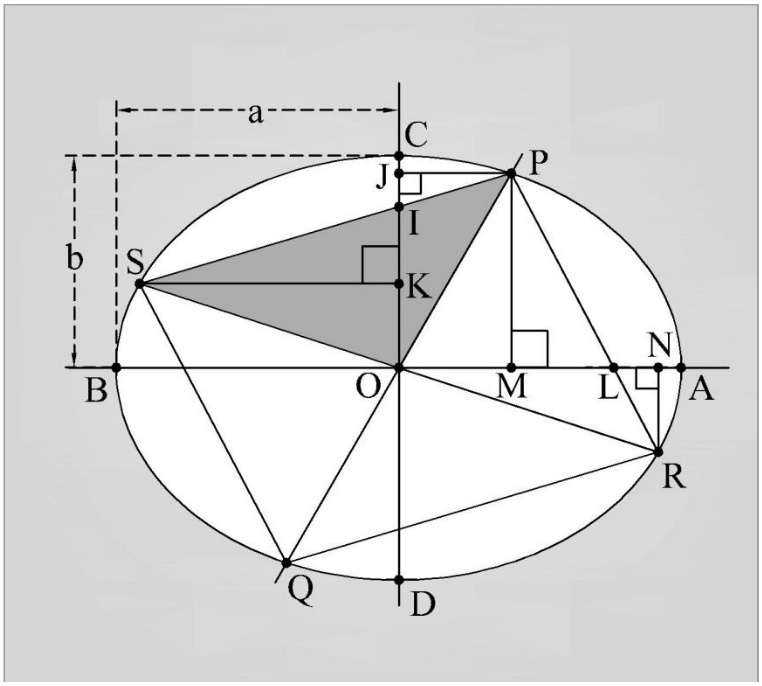

Figure 2: An Ellipse with the shaded area which is the area of triangle BOD

Therefore, $x_1 = a\cos\theta°$ – [1.11]
Therefore, $y_1 = b\sin\theta°$ – [1.12]
Let, point 'K' is the projection of point 'S' on minor axis.
$SK = ON = a\sin\theta°$, since $\angle ROA = \angle SOB$. Let, $SK = x_3$
Therefore, $x_3 = -a\sin\theta°$ – [1.13]
$RN = OK = b\cos\theta°$. Let, $OK = y_3$
Therefore, $y_3 = b\cos\theta°$ – [1.14]
We know that the Equation of straight line passing through any two points $(x_1, y_1)\&(x_3, y_3)$ is

$$y - y_1 = \left(\frac{y_3 - y_1}{x_3 - x_1}\right)(x - x_1) \quad \text{– –}[1.15]$$

Equation of line 'PS' passing through $P(a\cos\theta°, b\sin\theta°)$ and $S(-a\sin\theta°, b\cos\theta°)$

$$\Rightarrow \; y - b\sin\theta° = \left(\frac{b\cos\theta° - b\sin\theta°}{-a\sin\theta° - a\cos\theta°}\right)(x - a\cos\theta°) \quad \text{– – – – – – – – – – – – – – –}[1.16]$$

Referring figure 2, line PS is intersecting the minor axis at point I(x = 0). Therefore, substituting x = 0 in eqn.[1.16], we get

$$\Rightarrow \; y - b\sin\theta° = \left(\frac{b\cos\theta° - b\sin\theta°}{-a\sin\theta° - a\cos\theta°}\right)(0 - a\cos\theta°)$$

$$\Rightarrow \; y - b\sin\theta° = \frac{-b}{a}\left(\frac{\cos\theta° - \sin\theta°}{a\sin\theta° + a\cos\theta°}\right) \times (-a\cos\theta°)$$

$$\Rightarrow \; y - b\sin\theta° = \frac{b}{a}\left(\frac{\cos\theta° - \sin\theta°}{a\sin\theta° + a\cos\theta°}\right) \times (a\cos\theta°)$$

$$\Rightarrow\ y - b\sin\theta^\circ = b\cos\theta^\circ\left(\frac{\cos\theta^\circ - \sin\theta^\circ}{a\sin\theta^\circ + a\cos\theta^\circ}\right)$$

Therefore, $y = \left\{ b\cos\theta^\circ\left(\frac{\cos\theta^\circ - \sin\theta^\circ}{a\sin\theta^\circ + a\cos\theta^\circ}\right)\right\} + b\sin\theta^\circ$

Therefore, $y = \left\{\frac{b\cos^2\theta^\circ - b\sin\theta^\circ\cos\theta^\circ}{\sin\theta^\circ + \cos\theta^\circ}\right\} + b\sin\theta^\circ$

Therefore, $y = \left\{\frac{b\cos^2\theta^\circ - b\sin\theta^\circ\cos\theta^\circ + b\sin^2\theta^\circ + b\sin\theta^\circ\cos\theta^\circ}{\sin\theta^\circ + \cos\theta^\circ}\right\}$

Therefore, $y = b\left(\frac{\cos^2\theta^\circ + \sin^2\theta^\circ}{\sin\theta^\circ + \cos\theta^\circ}\right)$

Therefore, $y = \frac{b}{\sin\theta^\circ + \cos\theta^\circ}$

Therefore, $OI = \frac{b}{\sin\theta^\circ + \cos\theta^\circ}$ ———————————————————— [1.17]

Referring figure 2, Let Area of triangle $OKS = A_3$

Therefore, $A_3 = \frac{1}{2} \times OI \times SK$

Substituting, eqn.[1.17] & [1.12] in above equation, we get

$$A_3 = \frac{1}{2} \times \left(\frac{b}{\sin\theta^\circ + \cos\theta^\circ}\right) \times a\sin\theta^\circ$$

Therefore, $A_3 = \frac{ab\sin\theta^\circ}{2(\sin\theta^\circ + \cos\theta^\circ)}$ ————————————————————[1.18]

Referring figure 2, Let Area of triangle $OIP = A_4$

Therefore, $A_4 = \frac{1}{2} \times OI \times PJ$

Substituting, eqn.[1.17] & [1.13] in above equation, we get

Therefore, $A_4 = \frac{1}{2} \times \left(\frac{b}{\sin\theta^\circ + \cos\theta^\circ}\right) \times a\cos\theta^\circ$

Therefore, $A_4 = \frac{ab\cos\theta^\circ}{2(\sin\theta^\circ + \cos\theta^\circ)}$ ———————————————————— [1.19]

The total area of ΔOPS = Area of ΔOKP + Area of ΔOKS

$$\Delta OPS = A_3 + A_4$$

Substituting 1.18 and 1.19 in above, we get

$$\Delta OPS = \frac{ab\sin\theta^\circ}{2(\sin\theta^\circ + \cos\theta^\circ)} + \frac{ab\sin\theta^\circ}{2(\sin\theta^\circ + \cos\theta^\circ)}$$

$$\Delta OPS = \frac{ab(\sin\theta^\circ + \cos\theta^\circ)}{2(\sin\theta^\circ + \cos\theta^\circ)}$$

$\Delta OPS = \frac{ab}{2}$ ————————————————————————— [1.20]

The total area of parallelogram $PSQR = 2 \times (\text{Area of } \Delta OPR + \text{Area of } \Delta OPS)$ (Referring figure 3)

Substituting 1.10 and 1.20 in above, we get

$\therefore$ Area of parallelogram $PSQR = 2ab$ ———————————————[1.21]

$$\therefore \text{Area of parallelogram } PSQR = 2 \times \left(\frac{ab}{2} + \frac{ab}{2}\right)$$

The area of right angled triangle $AOC = \frac{1}{2} \times OA \times OC$

We know that $OA = a$ and $OC = b$ and substituting in above equation

The area of right angled triangle $\text{AOC} = \frac{1}{2} \times \text{a} \times \text{b}$ ———————————————— [1.22]

Referring figure 3, a rhombus is formed by connecting the extremities of major and minor axis. We know that in a rhombus, all four sides and all the internal vertex angles are equal. Therefore, AC = CB = BD = DA.

Therefore, the area of rhombus ABCD = ΔAOC +ΔBOC + ΔAOD + ΔBOD.

We now that $\Delta\text{AOC} = \Delta\text{BOC} = \Delta\text{AOD} = \Delta\text{BOD}$. Substituting equation 1.22 in above equation, we get

$$\text{Area of rhombus ABCD} = 4 \times \Delta\text{AOC} = \frac{1}{2} \times OA \times OC$$

$$\text{Area of rhombusABCD} = 4 \times \left(\frac{1}{2} \times \text{a} \times \text{b}\right)$$

$\text{Area of rhombus ABCD} = 2\text{ab}$ ———————————————————[1.23]

From equations [1.21] & [1.23], the Area of Parallelogram PSRQ = Area of Rhombus ADBC and the theorem is proved.

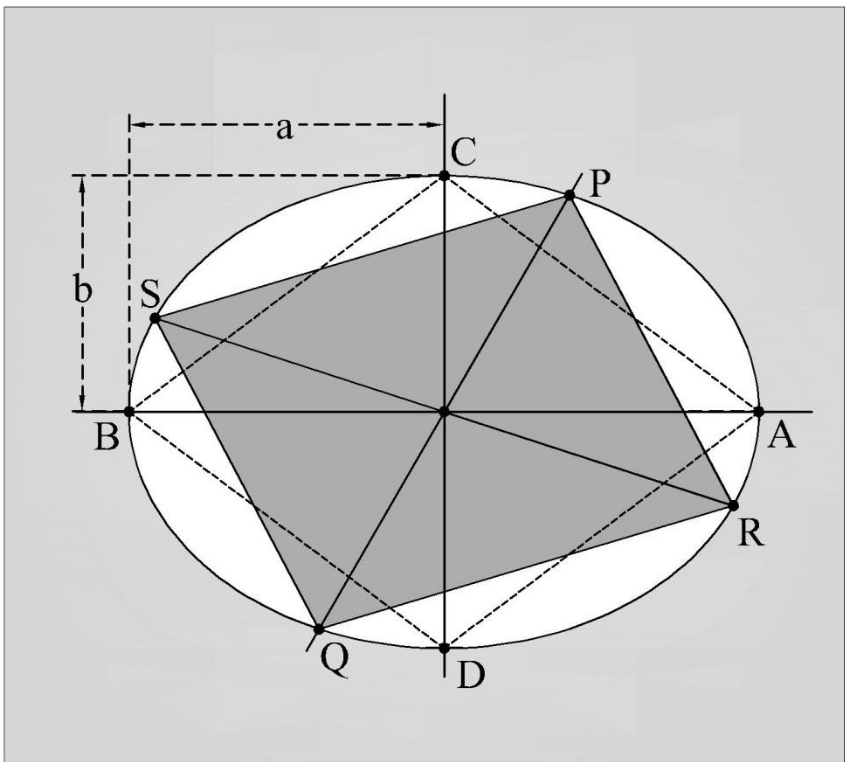

Figure 3: An Ellipse with Diameter PQ and Conjugate Diameter RS to PQ.
The shaded area which is the area of parallelogram by connecting PSQR

RESULTS AND DISCISSION

The theorem isverified with an example. An ellipse with the parameters ofmajor axis (a) = 4.0 units, minor axis (b) = 3.0 units and the diameters are drawn through point 'P' at ∠AOP = 0°, 15°, 30°, 45°, 60°, 75° and 90°. The pair conjugate diameters are also drawn to the corresponding diameters at specified angles. The dimensions of the parameters directly measured from the AutoCAD drawing corresponding to ∠AOP such that (i) PS = RQ, (ii) PR = SQ, (iii) PQ, (iv) RS, (v) Altitude 'h' and (vi) Area and (vii) Area of the parallelogram calculated by formula (i.e. Area = PS × h)are given in table-1 for comparison of the result of the theorem.

Table 1: Dimensions of the parameters of the parallelogram PSQR of an example

∠AOP	PS = RQ	PR = SQ	PQ	RS	Altitude (h)	Area of the parallelogram (Sq.units)	
						As per drawing	Usingformula
0°	5.0000	5.0000	8.0000	6.0000	4.8000	24.0000	24.0000
15°	5.4262	4.5354	7.7994	6.2585	4.4242	24.0000	24.0066
30°	5.6362	4.2702	7.3199	6.8131	4.2584	24.0000	24.0012
45°	5.6336	4.2764	6.7882	7.3430	4.2622	24.0000	24.0115
60°	5.4869	4.4603	6.3578	7.7187	4.3741	24.0000	24.0003
75°	5.2635	4.7218	6.0899	7.9318	4.5597	24.0000	24.0000
90°	5.0000	5.0000	6.0000	8.0000	4.8000	24.0000	24.0000

Conclusion

In this article the existing theorems have been briefly described in the introduction for the reference in order to understand the fundamental concept about the properties of the conjugate diameters. The author has attempted and developed two new theorems. These theorems have been derived mathematically, defined and proved with appropriate illustrations and examples for detailed explanation. These theorems define the mathematical relation between conjugate diameters and other parameters of the ellipse. These theorems may be helpful for those doing research or further study in the ellipse and geometry of conics.

REFERENCES

Bali NP (2005). Coordinate Geometry, (New Delhi, India: Laxmi Publications Private Ltd) **88** 341, 369, 398.

Christopher Clapham & James Nicolson (2009). The concise Oxford Dictionary of Mathematics (Oxford, UK, Oxford University Press) 257.

Weisstein Eric W (2003). CRC Concise Encyclopedia of Mathematics (New York: CRC Press of Wolfram Research Inc.) **2** 867, 1080, 2648, 1845, 1920, 412, 2932, 715 and 2148.