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Research Article

GEOMETRY OF SUBMERSIONS GENERATED BY KILLING VECTOR FIELDS

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ABSTRACT
In the paper I have studied the geometry of submersions over an orbit Killing vector fields. It is proved that there exists a Riemannian metric on the orbit with respect to which the submersion will be Riemannian, and foliation generated by submersion will be isoparametric.

Keywords: *Killing Vector Field, an Orbit, Submersion, Riemannian Submersion, Isoparametric Foliation, Principal Curvature, Sectional Curvature*

INTRODUCTION
Let M be a smooth Riemannian manifold of dimension n with the Riemannian metric g, ∇ - the Levi-Civita connection, $\langle\cdot,\cdot\rangle$ - inner product defined by the Riemannian metric g.
We denote by $V(M)$ the set of all smooth vector fields defined on M, through a $[X,Y]$ Lie bracket of vector fields $X, Y \in V(M)$. The set $V(M)$ is Lie algebra with Lie bracket.
Throughout the paper, the smoothness means smoothness of a class C^∞.
Definition 1: *Differentiable mapping $\pi: M \to B$ of a maximal rank, where B is smooth manifold of dimension m, $n > m$, is called submersion.*
By the theorem on the rank of a differentiable function for each point $p \in B$ the full inverse image $\pi^{-1}(p)$ is a submanifold of dimension $k = n - m$. Thus, submersion $\pi: M \to B$ generates a foliation F of dimension $k = n - m$, whose leaves are submanifolds $L_p = \pi^{-1}(p), p \in B$.
To the study of the geometry of submersions were devoted numerous papers (Zoyidov and Tursunov, 2015; Reinhart, 1959), in particular in paper O'Neil, (1996) derived the fundamental equations of submersion.
Let F be a foliation of dimension k, where $0 < k < n$ (Gromoll and Walschap, 2008). We denote by L_p leaf of foliation F, passing through a point $q \in M$, where $\pi(q) = p$, by T_qF tangent space of leaf L_p at the point $q \in L_p$, by $H(q)$ orthogonal complement of subspace T_qF.
As result arise sub bundle's $TF = \{T_qF\}$, $TH = \{H(q)\}$ of the tangent bundle TM and we have an orthogonal decomposition $TM = TF \oplus TH$.
Thus every vector field X is decomposable as: $X = X^v + X^h$, where $X^v \in TF$, $X^h \in TH$. If $X^h = 0$ (respectively $X^v = 0$), then the field X is called as vertical (respectively horizontal) vector field.
The submersion $\pi: M \to B$ is said to be Riemannian if differential preserves lengths of horizontal vectors. As it is known those Riemannian submersions generate Riemannian foliation (Reinhart, 1959).
We remark that foliation F is called Riemannian if every geodesic, orthogonal in some point to leaves, remains orthogonal to leaves in all points.
The curve is called as horizontal if it's tangential vector is horizontal.
Let $\gamma: [a,b] \to B$ is smooth curve in B, and $\gamma(a) = p$. Horizontal curve $\tilde{\gamma}: [a,b] \to M$, $\tilde{\gamma}(a) \in \pi^{-1}(p)$ is called as horizontal lift of a curve $\gamma: [a,b] \to B$, if $\pi\big(\tilde{\gamma}(t)\big) = \gamma(t)$ for all $t \in [a,b]$.
The map $S: V(F) \times H(F) \to V(F)$, defined by the formula $S(U,X) = \nabla_U^v X$, is called second basic tensor, where $V(F)$, $H(F)$ set of vertical and horizontal vector fields respectively.
At the fixed field of normal $X \in HF$, map $S(U,X)$ generates tensor field S_X of type (1,1):
$$S(U,X) = S_X U = \nabla_U^v X$$
Where, $\nabla_U^v X$ is vertical component of vector field $\nabla_U X$.
The tensor field S_X defines the bilinear form l_X:

$l_X(U,V) = \langle S_X U, V \rangle$.
The form $l_X(U,V)$ is called second basic form with respect to a normal vector field X.
The tensor field S_X is linear map and consequently it is defined by the matrix $S(U,X) = AU$.
Horizontal vector field X is called basic if vector field $[U,X]$ is also vertical for each vector field $U \in V(F)$. Eigenvalues of matrix A is called the principal curvature of foliation F, when vector field X is basic. If the principal curvatures are locally constant along leaf, then foliation F is called isoparametric.

Main Part

In this paper we study geometry of some submersions, which arise at study of geometry of Killing vector fields. Geometry of vector fields is subject of numerous studies in connection its importance in geometry and other areas of mathematics (Zoyidov and Tursunov, 2015; Narmanov and Saitova, 2014; Gromoll and Walschap, 2008).
Let's consider some set $D \subset V(M)$, which contains finite or infinite number of smooth vector fields. For a point $x \in M$ through $t \to X^t(x)$ we will denote the integral curve of a vector field X passing through a point x at $t = 0$. Map $t \to X^t(x)$ is defined in some domain $I(x) \subset R$, which generally depends on field X and point x.
Definition 2: *The orbit $L(x)$ of set D, passing through the point x, is defined as set of such points $y \in M$, such that there exists $t_i \in R$, and vector fields $X_i \in D$*

$$y = X_k^{t_k}\left(X_{k-1}^{t_{k-1}}(\dots(X_1^{t_1}(x))\dots)\right).$$

In Sussmann (1973) it is proved that each orbit of a set of smooth vector fields has a differential structure of the smooth immersed sub manifold of M.
Recall that the vector field X on M is called the Killing vector field, if the group of local transformations $x \to X^t(x)$ consists of isometries (Narmanov and Saitova, 2014).
Note that the Lie bracket of two fields of the field of Killing gives a field of Killing and a linear combination of Killing fields over the field of real numbers is also a field of Killing.
Therefore, the set of all Killing vector field on the manifold M, denoted $K(M)$, generates a Lie algebra over the field of real numbers. It is known that the Lie algebra $K(M)$ is finite-dimensional. We will denote through $A(D)$ the smallest Lie subalgebra of algebra $K(M)$, containing set D.
Since the algebra $K(M)$ finite, there exist vector fields $X_1, X_2, \dots, X_m$ that vectors $X_1(x), X_2(x), \dots, X_m(x)$ forms bases for the subspace $A_x(D)$ for each $x \in M$.
In Narmanov and Saitova (2014) proved the following theorem, which shows that each point in the orbit $L(x_0)$ can be reached from x_0 by finitely many "switches" with the use of the vector fields $X_1, X_2, \dots, X_m$ in a certain order.
***Theorem 1*:** *Set of points of form*

$$y = X_m^{t_m}\left(X_{m-1}^{t_{m-1}}(\dots(X_1^{t_1}(x_0))\dots)\right),$$

Where, $(t_1, t_2, \dots, t_m) \in R^m$, coincides with the orbit $L(x_0)$.
This theorem allows constructing various submersions $\pi: R^m \to L(x_0)$ using the vector fields $X_1, X_2, \dots, X_m$ by the formula

$$\pi(t_1, t_2, \dots, t_m) = X_m^{t_m}\left(X_{m-1}^{t_{m-1}}(\dots(X_1^{t_1}(x_0))\dots)\right).$$

Let's consider the Killing vector fields

$$Y_1 = \frac{\partial}{\partial x_1}, Y_2 = \frac{\partial}{\partial x_2}, Y_3 = -x_3\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_3}, Y_4 = -x_4\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_4}$$

on R^4. It is easy to check that the basis of sub algebra $A(D)$ consists of following vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = -x_3\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_3},$$
$$Y_4 = -x_4\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_4}, \quad X_5 = \frac{\partial}{\partial x_3}, \quad X_6 = \frac{\partial}{\partial x_4}$$

and consequently the orbit $L(p)$ for each point $p \in R^4$ coincides with space R^4.

We will define following submersion $\pi: R^6 \to R^4$ with formula

$$\pi(t_1, t_2, t_3, t_4, t_5, t_6) = X_4^{t_4}\left(X_6^{t_6}\left(X_2^{t_2}\left(X_3^{t_3}\left(X_5^{t_5}\left(X_1^{t_1}(O)\right)\right)\right)\right)\right),$$

Where, O - origin of coordinates in R^4.

Theorem 2: *There exists a Riemannian metric $\tilde{g}$ on R^4 that:*
a) Map $\pi: R^6 \to R^4$ is Riemannian submersion and it generates Riemannian foliation;
b) Submersion $\pi: R^6 \to R^4$ generates on R^6 a foliation of zero curvature;
c) Submersion $\pi: R^6 \to R^4$ generates on R^6 isoparametric foliation;
d) $(R^4, \tilde{g})$ is manifold of nonnegative curvature.

Proof. 1) Mapping π has the form

$$\pi(t_1, t_2, t_3, t_4, t_5, t_6) = (x_1, x_2, x_3, x_4)$$

where

$$\begin{cases} x_1 = t_1 \cos t_3 - t_5 \sin t_3, & x_2 = t_2 \cos t_4 - t_6 \sin t_4, \\ x_3 = t_1 \sin t_3 + t_5 \cos t_3, & x_3 = t_2 \sin t_4 + t_6 \cos t_4, \end{cases}$$

We show that the rank of the mapping $\pi: R^6 \to R^4$ at each point $q = (t_1, t_2, t_3, t_4, t_5, t_6)$ is equal 4.
The simple calculation shows, the Jacobi matrix of mapping π has the form:

$$J(\pi) = \begin{pmatrix} \cos t_3 & 0 & \sin t_3 & 0 \\ 0 & \cos t_4 & 0 & \sin t_4 \\ -t_1 \sin t_3 - t_5 \cos t_3 & 0 & t_1 \cos t_3 - t_5 \sin t_3 & 0 \\ 0 & -t_2 \sin t_4 - t_6 \cos t_4 & 0 & t_2 \cos t_4 - t_6 \sin t_4 \\ -\sin t_3 & 0 & \cos t_3 & 0 \\ 0 & -\sin t_4 & 0 & \cos t_4 \end{pmatrix}.$$

Since of each point p four of the six vectors $X_1(p)$, $X_2(p)$, $X_3(p)$, $X_4(p)$, $X_5(p)$, $X_6(p)$ linearly independent, the rank of the Jacobi matrix is equal four. Therefore, for each point $p = (x_1, x_2, x_3, x_4) \in R^4$ the full inverse image $\pi^{-1}(p)$ is a two-dimensional submanifold in R^6.
In our case for a point $p = (x_1, x_2, x_3, x_4) \in R^4$ the full inverse image $\pi^{-1}(p)$ has the form

$$\pi^{-1}(p) = L_p(u, v) = (t_1, t_2, t_3, t_4, t_5, t_6)$$

where

$$t_1 = x_1 \cos u + x_3 \sin u, \quad t_2 = x_2 \cos v + x_4 \sin v, \quad t_3 = u,$$
$$t_4 = v, \quad t_5 = -x_1 \sin u + x_3 \cos u, \quad t_6 = -x_2 \sin v + x_4 \cos v, \quad u, v \in R.$$

It is easy to check that the foliation F, generated by the submersion $\pi: R^6 \to R^4$, consists of two-dimensional surface in the R^6, and the vector-speeds of curves u and v on this surface (a vertical fields) has the form $U = \{t_5, 0, 1, 0, -t_1, 0\}$ and $V = \{0, t_6, 0, 1, 0, -t_2\}$ respectively.

This vector fields is a Killing field. Really, it is known that the vector field $X = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial t_i}$ in R^n is the vector field Killing if and only if the following conditions are satisfied (Narmanov and Saitova, 2014):

$$\frac{\partial \xi_i}{\partial t_j} + \frac{\partial \xi_j}{\partial t_i} = 0, \qquad i \neq j, \qquad \frac{\partial \xi_i}{\partial t_i} = 0, \qquad i = 1, \dots, n.$$

The vertical fields U, V satisfies these conditions and consequently is a Killing fields.
Thus foliation F is a Riemannian.
Let $\gamma: [a, b] \to R^4$, $\gamma(a) = p$ a smooth curve. Then for each point $q \in \pi^{-1}(p)$ there is it's horizontal lift $\tilde{\gamma}: [a, b] \to R^6$ such that $\tilde{\gamma}(a) = q$ (Zoyidov and Tursunov 2015).
Let X, Y vector fields on R^4, and X^*, Y^* - horizontal lifting of the vector fields, i.e. X^*, Y^* are horizontal vector fields on R^6 and $d\pi(X^*) = X$, $d\pi(Y^*) = Y$. Since the vector field $U = \{t_5, 0, 1, 0, -t_1, 0\}$ and $V = \{0, t_6, 0, 1, 0, -t_2\}$ are Killing fields, a inner product $\langle X^*, Y^* \rangle$ is constant along $L_p = \pi^{-1}(p)$ (O'Neil, 1996). Hence, if we will put $\langle X, Y \rangle(p) = \langle X^*, Y^* \rangle(q)$, where $q \in L_p$, $\langle X, Y \rangle$ is correctly defined inner

product, and we get Riemannian metric $\tilde{g}$ on R^4 Concerning this Riemannian metric submersion $\pi: R^6 \to R^4$ will be Riemannian.

2) We will calculate sectional curvature of manifold L_p, in the two-dimensional direction, defined by vertical vectors U, V at the point $q \in L_p \subset R^6$.

By the formula O'Neill (O'Neil, 1996), for the Riemannian submersion $\pi: M \to B$, sectional curvature $K, \widehat{K}$ of manifold M and the fiber L_p, are connected by the relation

$$K(U,V) = \widehat{K}(U,V) - \frac{\langle \nabla_U^h U, \nabla_V^h V\rangle - \left\|\nabla_U^h V\right\|^2}{\|U \wedge V\|^2},$$

Where, U and V are vertical vector fields of M, $U \wedge V$ – bivector constructed on vectors U, V. U^h - vertical complement of vector U.

As an Euclidean space R^6 is space of zero sectional curvature $K(U,V) = 0$ for any arbitrary two-dimensional direction. Therefore

$$\widehat{K}(U,V) = \frac{\langle \nabla_U^h U, \nabla_V^h V\rangle - \left\|\nabla_U^h V\right\|^2}{\|U \wedge V\|^2}.$$

The simple calculation that $\nabla_U^h V = 0$, $\langle \nabla_U^h U, \nabla_V^h V\rangle = 0$. Thus, $\widehat{K}(U,V) = 0$ and the manifold (L_p) is two-dimensional manifold of zero curvature.

3) Vector fields $H_1 = \{t_1, 0,0,0, t_5, 0\}$, $H_2 = \{0, t_2, 0,0,0, t_6\}$, $H_3 = \{-t_5, 0, t_1^2 + t_5^2, 0, t_1, 0\}$, $H_4 = \{0, -t_6, 0, t_2^2 + t_6^2, 0, t_2\}$ are basic fields, as:

$$[U, H_i] = 0, \qquad [V, H_i] = 0, \qquad i = \overline{1,4}.$$

We calculate the second fundamental tensor with respect to fields S_{H_i}, $i = \overline{1,4}$ corresponding second fundamental forms l_{H_i}, $i = \overline{1,4}$:

$$S_{H_1} U = \nabla_U H_1 = \{t_5, 0,0,0, -t_1, 0\}, \qquad S_{H_2} V = \nabla_U H_2 = \{0, t_6, 0,0,0, -t_2\},$$

respectively,

$$l_{H_1}(U,U) = \langle U, \nabla_U H_1\rangle = t_1^2 + t_5^2, \qquad l_{H_2}(V,V) = \langle V, \nabla_V H_2\rangle = t_2^2 + t_6^2,$$

and others forms are equal zero.

In this case eigen values λ_i corresponding matrixes A_i are equal:

$$\lambda_1 = \frac{\langle U, \nabla_U H_1\rangle}{U^2} = \frac{t_1^2 + t_5^2}{1 + t_1^2 + t_5^2}, \qquad \lambda_2 = \frac{\langle V, \nabla_V H_2\rangle}{V^2} = \frac{t_2^2 + t_6^2}{1 + t_2^2 + t_6^2},$$

respectively, others eigen values are equal zero.

It is easy to check that $U(\lambda_1) = 0$. Thus foliation F is isoparametric.

4) We will calculate sectional curvature of manifold $(R^4, \tilde{g})$ in the two-dimensional direction, defined by vectors U_q^*, V_q^* at the point $p \in R^4$.

By the formula O'Neill (1996), for the Riemannian submersion $\pi: M \to B$, sectional curvature K, K_* of manifolds M, B are connected by the relation

$$K(X,Y) = K_*(X^*, Y^*) - \frac{3}{4}\frac{|[X,Y]^v|^2}{||X \wedge Y||^2},$$

where X, Y are horizontal vector fields of M, $X \wedge Y$ – bivector constructed on vectors X, Y. X^v vertical complement of vector X.

As an Euclidean space R^6 is space of zero sectional curvature $K(X,Y) = 0$ for any arbitrary two-dimensional direction. Therefore,

$$K_*(X^*, Y^*) = \frac{3}{4}\frac{|[X,Y]^v|^2}{||X \wedge Y||^2}.$$

Thus, the manifold $(R^4, \tilde{g})$ is four-dimensional manifold of nonnegative curvature.

Now we can calculate sectional curvatures for the two dimensional directions defined by vector fields $H_1^* = d\pi(H_1)$, $H_2^* = d\pi(H_2)$, $H_3^* = d\pi(H_3)$, and $H_4^* = d\pi(H_4)$ on $(R^4, \tilde{g})$. By the fact that the mapping $\pi: R^6 \to R^4$ has maximum rank, vector fields H_1^*, H_2^*, H_3^*, and H_4^* are linearly independent in each point of manifold $(R^4, \tilde{g})$. In this case we can calculate sectional curvatures. As $K(H_i, H_j) = 0,\ i, j = \overline{1,4},\ i \neq j$, we will receive following expression for the curvature

$$K_*(H_1^*, H_3^*)(q) = \frac{3}{(1 + t_1^2 + t_5^2)^2}, \qquad K_*(H_2^*, H_4^*)(q) = \frac{3}{(1 + t_2^2 + t_6^2)^2},$$
$$K_*(H_1^*, H_2^*)(q) = K_*(H_1^*, H_4^*)(q) = K_*(H_2^*, H_3^*)(q) = K_*(H_3^*, H_4^*)(q) = 0.$$

The theorem 2 is proved.

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