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Research Article

DOUBLE-RULE EXTRACTION BASED ON OBJECT-INDUCED THREE-WAY CONCEPT LATTICE

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ABSTRACT
Three-way concept lattice proposed recently is an extension of classical concept lattice in formal concept analysis. This paper presents a positive answer to the open problem proposed by Liu *et al.*, (2016). The open problem is: how to extract rule in the decision formal context based on object-induced three-way concept lattice.

Keywords: *Three-Way Concept Analysis, Rule Extraction, Decision Context*

INTRODUCTION
Formal concept analysis (*FCA*), which proposed by Ganter and Wille, (1999) contains two key problems: one is to find formal concept and another is to construct concept lattice. Three-way concept analysis (3*WCA*) combined *FCA* with three-way decision (3*WD*) and proposed by Qi *et al.*, (2014). Three-way concept is also determined by extension and intension. The difference is the extension (intension) of three-way concept contains two parts. The two parts in 3*WCA* present the meaning of 'jointly possessed' and 'jointly not possessed'. Recently, many scholars have made some results on 3*WCA*, such as the algorithm of constructing three-way concept lattice and the attribute reduction of three-way concept lattice. In order to handle uncertainty and ambiguity of knowledge, the current research is mainly focus on the 3*WCA* under the incomplete or fuzzy formal context.
Liu *et al.*, (2016) provided an open problem that how to extract rule in decision formal context based on object-induced three-way concept lattice. This paper will give a positive answer for this open problem.
The construction of this paper is as follows. At first, we concisely review the theoretical knowledge of *FCA* and 3*WCA*. Then, we will discuss the relationship between object-induced three-way concept lattices. The notions of the positive-rule and negative-rule and double-rule based on object-induced three-way concept lattice are introduced. Furthermore, we give the idea of extracting double-rule from object-induced three-way concept lattice and the effectiveness of this idea is illustrated by an example.

Preliminaries
This section will recall some preliminaries works for *FCA* and 3*WCA*. For more detail, *FCA* is referred to Ganter and Wille, (1999) and 3*WCA* is seen (Qi *et al.*, 2014).
Definition 1 (1) (Ganter and Wille, 1999) Let (G, M, I) be formal context. For $X \subseteq G$ and $A \subseteq M$, the positive operators,*: $\rho(G) \to \rho(M)$ and *: $\rho(M) \to \rho(G)$, are defined by $X^* = \{m \in M | \forall x \in X, xIm\}$ and $A^* = \{x \in G | \forall m \in A, xIm\}$.
(2) (Qi *et al.*, 2014) Let (G, M, I) be formal context. For $X \subseteq G$ and $A \subseteq M$, the negative operators, $\bar{*}$: $\rho(G) \to \rho(M)$ and $\bar{*}$: $\rho(M) \to \rho(G)$, are defined by $X^{\bar{*}} = \{m \in M | \forall x \in X, xI^c m\}$ and $A^{\bar{*}} = \{x \in G | \forall m \in A, xI^c m\}$. Here, $I^c = (G \times M) - I$.
The concept lattice determined by (*, *) is denoted by $L(G, M, I)$. The concept lattice determined by ($\bar{*}$, $\bar{*}$) is denoted by N$L(G, M, I)$.
(3) (Qi *et al.*, 2014) Let (G, M, I) be formal context. For $X \subseteq G$ and $A, B \subseteq M$, the object-induced three-way operators, $\lessdot: \rho(G) \to \rho(M) \times \rho(M)$ and $\gtrdot: \rho(M) \times \rho(M) \to \rho(G)$ are defined by

$$X^{\lessdot} = (X^*, X^{\bar{*}}),\ (A, B)^{\gtrdot} = \left\{x \in G \middle| x \in A^* \ and \ x \in B^{\bar{*}}\right\} = A^* \cap B^{\bar{*}}.$$

The object-induced three-way concept (*OE*-concept) is a pair $(X,(A,B))$ with $X^{\lessdot}=(A,B)$ and $(A,B)^{\gtrdot}=X$. The set of all *OE*-concept is denoted by $OEL(G,M,I)$.

Definition 2 (1) (Zhang *et al.*, 2005) Let $L(G,M,I)$ and $L(G,N,J)$ be concept lattice. If for any $(Y,B)\in L(G,N,J)$, there exists $(X,A)\in L(G,M,I)$ such that $X=Y$. Then $L(G,M,I)$ is finer than $L(G,N,J)$ and denoted by $L(G,M,I)\leq L(G,N,J)$.

(2) (Wei *et al.*, 2008) Let $K=(G,M,I,N,J)$ be decision formal context where *M* is conditional attribute set and *N* is decision attribute set. If $L(G,M,I)\leq L(G,N,J)$, then $K=(G,M,I,N,J)$ is consistent.

Algorithm

The following mainly describes the idea of rule extraction based on object-induced three-way concept lattice in the decision formal context. At first, the partial order between object-induced three-way concept lattices is introduced.

Definition 3 (1) Let $OEL(G,M,I)$ and $OEL(G,N,J)$ be object-induced three-way concept lattice. For any $(Y,(C,D))\in OEL(G,N,J)$, there exists $(X,(A,B))\in OEL(G,M,I)$ such that *X*=*Y*, then *OEL*(*G*, *M*, *I*) is finer than *OEL*(*G*, *N*, *J*) and denoted by $OEL(G,M,I)\leq OEL(G,N,J)$.

(2) Let $K=(G,M,I,N,J)$ be decision formal context. If $OEL(G,M,I)\leq OEL(G,N,J)$, then $K=(G,M,I,N,J)$ is consistent based on object-induced three-way concept lattice.

Next, the difference and relationship between the two kinds of consistency are discussed as follows.

Theorem 1: Let $K=(G,M,I,N,J)$ be decision formal context. If $OEL(G,M,I)\leq OEL(G,N,J)$, then $L(G,M,I)\leq L(G,N,J)$.

Proof: For any $(Y,B)\in L(G,N,J)$, we can get $(Y,(B,\bar{Y}^{*}))\in OEL(G,N,J)$ according to the definition of *OE*-concept.

Since $OEL(G,M,I)\leq OEL(G,N,J)$, there exists *OE*-concept $(Y,(C,\bar{Y}^{*}))\in OEL(G,M,I)$. Therefore, $(Y,Y^{*})\in L(G,M,I)$. Based on Definition 2(1), we can obtain $L(G,M,I)\leq L(G,N,J)$.

We will give an example to show the incorrect of the converse of Theorem 1.

Example 1

The decision formal context $K=(G,M_1,I_1,N_1,J_1)$ with $G=\{1,2,3,4\}$ and conditional attribute set $M_1=\{a,b,c,d\}$ and decision attribute set $N_1=\{g,h,k\}$ is shown as Table 1. Given space limitations, we omit the representations of $OEL(G,M_1,I_1)$ and $OEL(G,N_1,J_1)$.

Table 1: A Decision Formal Context $K=(G,M_1,I_1,N_1,J_1)$

	a	**b**	**c**	**d**	**g**	**h**	**k**
1	×	×			×	×	
2	×					×	
3		×	×	×	×		
4	×			×		×	×

Here, $L(G,M_1,I_1)\leq L(G,N_1,J_1)$. But the conclusion $OEL(G,M_1,I_1)\leq OEL(G,N_1,J_1)$ does not hold. For $(123,(\emptyset,k))\in OEL(G,N_1,J_1)$, there is no *OE*-concept in $OEL(G,M_1,I_1)$, whose extension is equal to $\{1,2,3\}$. Hence, the decision formal context $K=(G,M_1,I_1,N_1,J_1)$ is not consistent based on object-induced three-way concept lattice.

Compared with the general sense of consistency, the above discuss shows that decision formal context based on object-induce three-way concept lattice is more consistent.

In what follows, we present the definitions of positive-rule and negative-rule and double-rule.

Definition 4: (1) Let $L(G,M,I,N,J)$ be consistent based on object-induced three-way concept lattice. For

$(X,A) \in L(G,M,I)$ and$(Y,B) \in L(G,N,J)$, the positive-rule $A \to B$ such that $X \subseteq Y$. Analogously, for $(Z,C) \in NL(G,M,I)$ and $(W,D) \in NL(G,N,J)$, the negative-rule $C \to D$ such that $Z \subseteq W$.
(2) Let $A_1 \to B_1$and $A_2 \to B_2$ be positive-rule (negative-rule). If $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$, then, $A_2 \to B_2$ can be derived from $A_1 \to B_1$. $A_2 \to B_2$ is called redundant positive-rule (negative-rule).

The positive-rule (negative-rule) $A \to B$ express that all objects that do (not) possess attribute set *A* must (not) possess attribute set *B*.

Definition 5: (1) Let $L(G,M,I,N,J)$ be consistent based on object-induced three-way concept lattice. For $(X,(A,B)) \in OEL(G,M,I)$ and $(Y,(C,D)) \in OEL(G,N,J)$, if $X \subseteq Y$ and $X \neq \emptyset, G$ and $Y \neq \emptyset, G$, then, $(A,B) \to (C,D)$ is called double-rule.
(2) Let $(A_1,B_1) \to (C_1,D_1)$ and $(A_2,B_2) \to (C_2,D_2)$ be double-rule. If $(A_1,B_1) \subseteq (A_2,B_2)$ and $(C_2,D_2) \subseteq (C_1,D_1)$, then $(A_2,B_2) \to (C_2,D_2)$ can be derived from $(A_1,B_1) \to (C_1,D_1)$. $(A_2,B_2) \to (C_2,D_2)$ is called redundant double-rule.

Double-rule $(A,B) \to (C,D)$ express that all objects which possess attribute set *A* and do not possess any attribute of *B* must possess attribute set *C* and do not possess any attribute of *D*.

For the convenience of the following discussion, we note the set of non-redundant set of positive-rule (negative-rule) for $R^{++}(R^{--})$. Note the set of non-redundant set of double-rule for R^*.

The relationship between these rules is as follows.

Theorem 2: Let $L(G,M,I,N,J)$ be consistent based on object-induced three-way concept lattice. $(A,B) \to (C,D)$ is a double-rule if and only if $A \cap B = \emptyset$ and $A \to C \in R^+$ and $B \to D \in R^-$.

Proof: Suppose $(X,A) \in L(G,M,I)$, $(Y,C) \in L(G,N,J)$, $(Z,B) \in NL(G,M,I)$ and $(W,D) \in NL(G,N,J)$. When $(A,B) \to (C,D)$ is a double-rule, by the Definition 5(1), $A \to C \in R^+$ and $B \to D \in R^-$ obviously holds.

When $A \cap B = \emptyset$ and $A \to C \in R^+$ and $B \to D \in R^-$, since $A \to C \in R^+$ and $B \to D \in R^-$, we have $X \subseteq Y$ and $Z \subseteq W$. There must exists $(X \cap Z,(A,B)) \in OEL(G,M,I)$ and $(Y \cap W,(C,D)) \in OEL(G,N,J)$ such that $X \cap Z \subseteq Y \cap W$. According to Definition 5(1), $(A,B) \to (C,D)$ is a double-rule of $K = (G,M,I,N,J)$.

Theorem 3: Let $L(G,M,I,N,J)$ be consistent based on object-induced three-way concept lattice. $(A,B) \to (C,D)$ is a non-redundant double-rule if and only if $A \cap B = \emptyset$ and $A \to C \in R^{++}$ and $B \to D \in R^{--}$

Proof: The proof of the conclusion is omitted because the method is similar to Theorem 2.

Main Ideas

We consider only the non-redundant double-rule extraction based on object-induced three-way concept lattice in decision formal context. Because positive-rule and negative-rule can be seen as decision rule in the general sense. Hence, the extraction of positive-rule and negative-rule can be obtained by some existing algorithms. For example, the reader can please refer to He *et al.*, (2009). In this paper, we only introduce the idea of extracting non-redundant double-rule.

Generate Double-Rule

Input: decision formal context $K = (G,M,I,N,J)$
Output: the set R^* of non-redundant double-rule

Step 1 initialize $R^* = \emptyset$;
Step 2 generate the set R^{++} of non-redundant positive-rule;
Step 3 generate the set R^{--} of non-redundant negative-rule;
Step 4 for every $A \to C \in R^{++}$ and $B \to D \in R^{--}$ with $A \cap B = \emptyset$
$R^* = R^* \cup \{(A,B) \to (C,D)\}$;
Step 5 output R^*.

Example 2

An example is used to illustrate the effectiveness of this idea.

The decision formal context $K = (G,M,I,N,J)$ with $G = \{1,2,3,4\}$ and conditional attribute set $M =$

$\{a,b,c,d,e,f\}$ and decision attribute set $N=\{g,h,k\}$ is shown as Table 3. $OEL(G,M,I)$ and $OEL(G,N,J)$ are shown as Figure 1 and Figure 2, respectively.

Table 2: A Decision Formal Context $K=(G,M,I,N,J)$

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*k*
1			×			×			×
2				×	×	×	×		×
3	×	×	×		×	×		×	×
4				×	×		×		

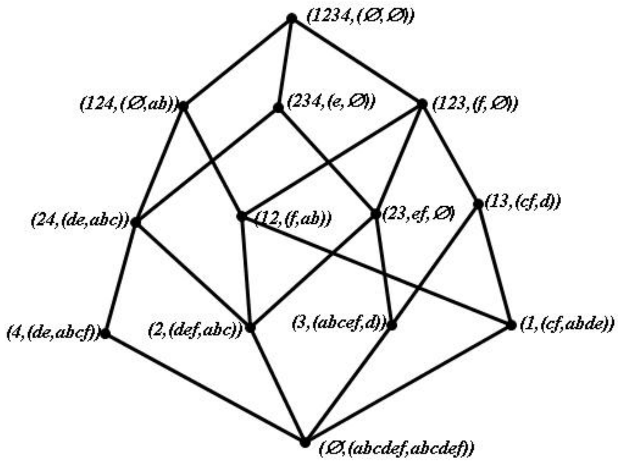

Figure 1: $OEL(G,M,I)$

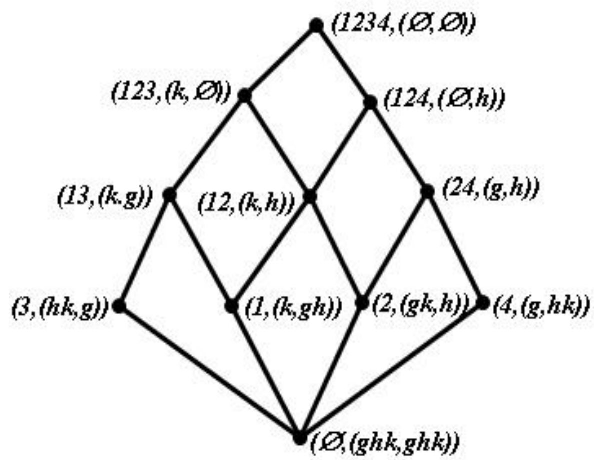

Figure 2: $OEL(G,N,J)$

The sets of non-redundant positive-rule and negative-rule and the non-redundant double-rule are shown as Table 3.

Table 3: Comparison of Three Rules (R^{++},R^{--},R^{*})

R^{++}	R^{--}	R^{*}
f→*k*	*d*→*g*	(*f*,*d*) →(*k*,*g*)
de→*g*	*ab*→*h*	(*f*,*ab*) →(*k*,*h*)
abcef→*hk*	*abde*→*gh*	(*de*,*ab*) →(*g*,*h*)
def→*gk*	*abcf*→*hk*	(*abcef*,*d*) →(*hk*,*g*)
		(*f*,*abde*) →(*k*,*gh*)
		(*def*,*ab*) →(*gk*,*h*)
		(*de*,*abcf*) →(*g*,*hk*)

Example 2 shows the relationship and difference of these rules. Therefore, we can quickly find the non-redundant double-rule by the non-redundant positive-rule and negative-rule.

Conclusion

In this paper, we present an idea of extracting decision rule based on object-induced three-way concept lattice in decision formal context. Then, we answer the open problem in Liu *et al.*, (2016).

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REFERENCES

Ganter B and Wille R (1999). *Formal Concept Analysis: Mathematical Foundations,* (Springer-Verlag Berlin, Heidelberg, Germany).

He XW, Niu HF, Xu LM and Ma Y (2009). Decision rules extraction based on trend concept lattice. *Journal of Computer Applications* **29**(4) 1106-1109.

Li MZ and Wang GY (2015). Approximate concept construction with three-way decisions and attribute reduction in incomplete contexts. *Knowledge-Based Systems* **91** 165-178.

Liu L, Qian T and Wei L (2016). Rules extraction in formal decision contexts based on attribute-Induced three-way concept lattice. *Journal of Northwest University (Natural Science Edition)* **46**(4) 481-487.

Qi JJ, Qian T and Wei L (2016). The connections between three-way and classical concept lattices. *Knowledge-Based Systems* **91**(C) 143-151.

Qi JJ, Wei L and Yao YY (2014). Three-way formal concept analysis. *Rough Sets and Knowledge Technology* **8818** 732-741.

Singh PK (2016). Three-way fuzzy concept lattice representation using neutrosophic set. *International Journal of Machine Learning and Cybernetics* 1-11.

Wei L, Qi JJ and Zhang WX (2008). Attribute reduction theory of concept lattice based on decision formal contexts. *Science in China Series F: Information Sciences* **51**(7) 910-923.

Yao YY (2016). Interval sets and three-way concept analysis in incomplete contexts. *International Journal of Machine Learning and Cybernetics* 1-18.

Zhang WX, Wei L and Qi JJ (2005). Attribute reduction theory and approach to concept lattice. *Science in China Series F Information Sciences* **48**(6) 713-726.