International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2016 Vol. 6 (4) October-December, pp. 27-35/Prasad
Research Article

STABILITY ANALYSIS OF A THREE SPECIES SYN-ECO-SYSTEM WITH MORTALITY RATES FOR THE FIRST AND THIRD SPECIES

***B. Hari Prasad**
Department of Mathematics, Chaitanya Degree College (Autonomous), Hanamkonda, Telangana State, India-506 001
**Author for Correspondence*

ABSTRACT
In this paper, the system comprises of a commensal (S_1) common to two hosts S_2 and S_3 with mortality rate for the two species S_1 and S_3. Here all the three species posses limited resources. The model equations constitute a set of three first order non-linear simultaneous differential equations. Criteria for the asymptotic stability of all the eight equilibrium states are established. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. Trajectories of the perturbations over the equilibrium states are illustrated. Further the global stability of the system is established with the aid of suitably constructed Liapunov's functions.

Keywords**:** *Asymptotically Stable, Commensal, Equilibrium State, Host, Liapunov's Function, Stable, Trajectories, Unstable*

INTRODUCTION
Ecology is a branch of life sciences connected to the existence of diverse species in the same environment and habitat. It is natural that two or more species living in a common habitat interact in different ways. Significant research in the area of theoretical ecology has been thresholded by Lotka (1925) and Volterra (1931).

Several mathematicians and ecologists contributed to the growth of this area of knowledge. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation and so on. Mathematical Modeling is a vital role in providing insight in to the mutual relationships between the interacting species. The general concepts of modeling have been discussed by several authors Colinvaux (1986), Kapur (1985), Kushing (1977), Meyer (1985). Srinivas (1991) studied competitive ecosystem of two species and three species with limited and unlimited resources. Later, Narayan and Charyulu (2007) studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Stability analysis of competitive species was carried out by Reddy *et al.,* (2007), Sharma and Charyulu (2008), while Ravindra Reddy (2008) investigated mutualism between two species. Acharyulu and Charyulu (2011) derived some productive results on various mathematical models of ecological Ammensalism with multifarious resources in the manifold directions. Further, Kumar (2010) derived some mathematical models of ecological commensalism. The present author Prasad (2014; 2015; 2016) studied continuous and discrete models on the three species syn-ecosystems. The present investigation is an analytical study of three species (S_1, S_2, S_3) syn-eco system with mortality rate for the three species. The system comprises of two hosts S_1, S_2 and one commensal S_3 i.e., S_1 and S_2 both benefit S_3, without getting themselves affected either positively or adversely. Further, S_1 and S_2 are neutral. Commensalism is a symbiotic interaction between two populations where one population (S_1) gets benefit from (S_2) while the other (S_2) is neither harmed nor benefited due to the interaction with (S_1). The benefited species (S_1) is called the commensal and the other (S_2) is called the host. A real-life example of commensalism is a squirrel in an oak tree gets a place to live and food for its survival, while the tree remains neither benefited nor harmed.

Notations and Basic Equations

$N_i(t)$: The population strength of S_i at time t, $i = 1, 2, 3$

t : Time instant

d_i	: Natural death rate of S_i, $i=1,3$
g_2	: Natural growth rate of S_2
a_{ii}	: Self inhibition coefficients of S_i, $i=1,2,3$
a_{12}, a_{13}, a_{23}	: Interaction coefficients
$e_i = \frac{d_i}{a_{ii}}$	: Extinction coefficient of S_i, $i=1,3$
$k_2 = \frac{g_2}{a_{22}}$	: Carrying capacities of S_2

Further, the variables N_1, N_2, N_3 are non-negative and the model parameters $d_1, g_2, d_3, a_{11}, a_{12}, a_{13}$, a_{22}, a_{33}, a_{23} are assumed to be non-negative constants.

The model equations for syn ecosystem is given by the following system of first order non-linear ordinary differential equations.

Equation for the first species (N_1):

$$\frac{dN_1}{dt} = \left(-d_1 - a_{11}N_1 + a_{12}N_2 + a_{13}N_3\right)N_1 \tag{1}$$

Equation for the second species (N_2):

$$\frac{dN_2}{dt} = \left(g_2 - a_{22}N_2 + a_{23}N_3\right)N_2 \tag{2}$$

Equation for the third species (N_3):

$$\frac{dN_3}{dt} = -\left(d_3 + a_{33}N_3\right)N_3 \tag{3}$$

Equilibrium States

The system under investigation has eight equilibrium states given by

$$\frac{dN_i}{dt} = 0, i = 1, 2, 3 \tag{4}$$

Fully washed out state.

$E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$

States in which only two of the tree species are washed out while the other one is not.

$E_2 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = -e_3$

$E_3 : \bar{N}_1 = 0, \bar{N}_2 = k_2, \bar{N}_3 = 0$

$E_4 : \bar{N}_1 = -e_1, \bar{N}_2 = 0, \bar{N}_3 = 0$

States in which only one of the tree species is washed out while the other two are not.

$E_5 : \bar{N}_1 = 0, \bar{N}_2 = k_2 - \frac{a_{23}e_3}{a_{22}}, \bar{N}_3 = -e_3$

$E_6 : \bar{N}_1 = -\left(e_1 + \frac{a_{13}e_3}{a_{11}}\right), \bar{N}_2 = 0, \bar{N}_3 = -e_3$

$E_7 : \bar{N}_1 = \frac{a_{12}k_2}{a_{11}} - e_1, \bar{N}_2 = k_2, \bar{N}_3 = 0$

The co-existent state (or) normal steady state.

$$E_8 : \bar{N}_1 = \frac{1}{a_{11}}\left[a_{12}k_2 - \left(\frac{a_{12}a_{23}e_3}{a_{22}} + a_{13}e_3 + d_1\right)\right], \bar{N}_2 = k_2 - \frac{a_{23}e_3}{a_{22}}, \bar{N}_3 = -e_3$$

Stability Analysis of Equilibrium States

Let us consider small deviations from the steady state

i.e., $N_i(t) = \bar{N}_i + u_i(t), i = 1,2,3$ (5)

where $u_i(t)$ is a small perturbations in the species S_i.

The basic equations are linearized over the equilibrium state $\bar{N} = \left(\bar{N}_1, \bar{N}_2, \bar{N}_3\right)$ to obtain the equations for the perturbed state as

$$\frac{dU}{dt} = AU \quad (6)$$

with

$$A = \begin{bmatrix} -d_1 - 2a_{11}\bar{N}_1 + a_{12}\bar{N}_2 + a_{13}\bar{N}_3 & a_{12}\bar{N}_1 & a_{13}\bar{N}_1 \\ 0 & g_2 - 2a_{22}\bar{N}_2 + a_{23}\bar{N}_3 & a_{23}\bar{N}_2 \\ 0 & 0 & -d_3 - 2a_{33}\bar{N}_3 \end{bmatrix} \quad (7)$$

The characteristic equation for the system is given by

$$|A - \lambda I| = 0 \quad (8)$$

The equilibrium state is stable, if all the roots of the equation (8) are negative in case they are real or have negative real parts, in case they are complex.

Stability of $E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$

In this case, we have the characteristic equation as

$$\begin{vmatrix} -d_1 - \lambda & 0 & 0 \\ 0 & g_2 - \lambda & 0 \\ 0 & 0 & -d_3 - \lambda \end{vmatrix} = 0 \quad (9)$$

The characteristic roots of (9) are $-d_1$, g_2 and $-d_3$. Since one of these three roots is positive. Hence, the state is unstable and the solutions of the equations (6) are

$$u_i = u_{i0}\, e^{-d_i t};\; u_2 = u_{20}\, e^{g_2 t},\; i = 1,3 \quad (10)$$

where u_{10}, u_{20}, u_{30} are the initial values of u_1, u_2, u_3 respectively.

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$$\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{d_1}} = \left(\frac{u_2}{u_{20}}\right)^{-\frac{1}{g_2}} = \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{d_3}} \quad (11)$$

Stability of $E_2 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = -e_3$

In this case, we have

$$\begin{vmatrix} -(d_1 + a_{13}e_3) - \lambda & 0 & 0 \\ 0 & (g_2 - a_{23}e_3) - \lambda & 0 \\ 0 & 0 & d_3 - \lambda \end{vmatrix} = 0 \quad (12)$$

The characteristic roots are $-(d_1+a_{13}e_3)$, $(g_2-a_{23}e_3)$ and d_3. Since one of these three roots is positive, hence, the state is unstable and the solutions are

$$u_1 = u_{10}e^{-(d_1+a_{13}e_3)t};\, u_2 = u_{20}e^{(g_2-a_{23}e_3)\,t};\, u_3 = u_{30}e^{d_3 t} \tag{13}$$

The trajectories in the u_1-u_2 and u_2-u_3 planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{-\frac{1}{(d_1+a_{13}e_3)}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{(g_2-a_{23}e_3)}} = \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{d_3}}$$

Stability of $E_3 : \bar{N}_1 = 0,\ \bar{N}_2 = k_2,\ \bar{N}_3 = 0$

In this case, we get

$$\begin{vmatrix} (a_{12}k_2-d_1)-\lambda & 0 & 0 \\ 0 & -g_2-\lambda & a_{23}k_2 \\ 0 & 0 & -d_3-\lambda \end{vmatrix} = 0 \tag{14}$$

The characteristic roots are $d_1-a_{12}k_2$, $-g_2$ and $-d_3$.

Case I: When $a_{12}k_2-d_1<0$

In this case all the three roots are negative, hence, the state is stable. The equations (6) yield the solutions,

$$u_1 = u_{10}e^{-(d_1+a_{12}k_2)t};\, u_2 = (u_{20}-\psi)e^{-g_2 t}+\psi e^{-a_3 t};\, u_3 = u_{30}e^{-d_3 t} \tag{15}$$

where

$$\psi = \frac{a_{23}k_2u_{30}}{g_2-d_3};\ g_2 \neq d_3 \tag{16}$$

It can be noticed that $u_1 \to 0, u_2 \to 0$ and $u_3 \to 0$ as $t \to \infty$

Case II: When $a_{12}k_2-d_1=0$

In this case the state is neutrally stable and the solution curves of (6) are given by

$$u_1 = u_{10};\, u_2 = (u_{20}-\psi)e^{-g_2 t}+\psi e^{-a_3 t};\, u_3 = u_{30}e^{-d_3 t} \tag{17}$$

Case III: When $a_{12}k_2-d_1>0$

In this case the state is unstable and the solution curves of (6) are given by

$$u_1 = u_{10}e^{(d_1+a_{12}k_2)t};\, u_2 = (u_{20}-\psi)e^{-g_2 t}+\psi e^{-a_3 t};\, u_3 = u_{30}e^{-d_3 t} \tag{18}$$

The trajectories in the u_1-u_2 and u_2-u_3 planes are

$$u_2 = (u_{20}-\psi)\left(\frac{u_1}{u_{10}}\right)^{\frac{g_2}{d_1-a_{12}k_2}} + \psi\left(\frac{u_1}{u_{10}}\right)^{\frac{d_3}{d_1-a_{12}k_2}};\ u_2 = (u_{20}-\psi)\left(\frac{u_3}{u_{30}}\right)^{\frac{g_2}{a_3}} + \frac{u_3\psi}{u_{30}} \tag{19}$$

Stability of $E_4 : \bar{N}_1 = -e_1,\ \bar{N}_2 = 0,\ \bar{N}_3 = 0$

In this state, we have

$$\begin{vmatrix} d_1-\lambda & -a_{12}e_1 & -a_{13}e_1 \\ 0 & g_2-\lambda & 0 \\ 0 & 0 & -d_3-\lambda \end{vmatrix} = 0 \tag{20}$$

The characteristic roots are d_1, g_2 and $-d_3$. Since two of these three roots are positive, hence, the state is unstable. The equations (6) yield the solutions,

$$u_1 = (u_{10} - \psi_1 - \psi_2)e^{d_1 t} + \psi_1 e^{g_2 t} + \psi_2 e^{-d_3 t}; u_2 = u_{20}e^{g_2 t}; u_3 = u_{30}e^{-d_3 t} \quad (21)$$

Where

$$\psi_1 = \frac{a_{12}e_1 u_{20}}{d_1 - g_2}\ ;\ \psi_2 = \frac{a_{13}e_1 u_{30}}{d_1 + d_3};\ d_1 \neq g_2 \quad (22)$$

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_1 = (u_{20} - \psi_1 - \psi_2)\left(\frac{u_2}{u_{20}}\right)^{\frac{d_1}{g_2}} + \frac{u_2\psi_1}{u_{20}} + \psi_2\left(\frac{u_2}{u_{20}}\right)^{-\frac{d_3}{g_2}}\ ;\ \left(\frac{u_2}{u_{20}}\right)^{-d_3} = \left(\frac{u_3}{u_{30}}\right)^{g_2} \quad (23)$$

Stability of $E_5 : \bar{N}_1 = 0,\ \bar{N}_2 = k_2 - \frac{a_{23}e_3}{a_{22}},\ \bar{N}_3 = -e_3$

In this case, we have

$$\begin{vmatrix} \alpha_1 - \lambda & 0 & 0 \\ 0 & (a_{23}e_3 - g_2) - \lambda & \frac{a_{23}}{a_{22}}(g_2 - a_{23}e_3) \\ 0 & 0 & d_3 - \lambda \end{vmatrix} = 0 \quad (24)$$

where

$$\alpha_1 = \frac{a_{12}}{a_{22}}(g_2 - a_{23}e_3) - (d_1 + a_{13}e_3) \quad (25)$$

The characteristic roots are $\alpha_1,\ a_{23}e_3 - g_2$ and d_3. Since one of these three roots is positive, hence, the state is unstable. The equations (6) yield the solutions,

$$u_1 = u_{10}e^{-\alpha_1 t}; u_2 = (u_{10} - \chi)e^{(a_{23}e_3 - g_2)t} + \chi e^{d_3 t}; u_3 = u_{30}e^{d_3 t} \quad (26)$$

where

$$\chi = \frac{a_{23}(g_2 - a_{23}e_3)u_{30}}{a_{22}(d_3 + g_2 - a_{23}e_3)};\ \text{with } d_3 + g_2 \neq a_{23}e_3 \quad (27)$$

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_2 = (u_{20} - \chi)\left(\frac{u_1}{u_{10}}\right)^{\frac{a_{23}e_3 - g_2}{\alpha_1}} + \chi\left(\frac{u_1}{u_{10}}\right)^{\frac{d_3}{\alpha_1}}\ ;\ u_2 = (u_{20} - \chi)\left(\frac{u_3}{u_{30}}\right)^{\frac{a_{23}e_3 - g_2}{d_3}} + \frac{u_3\chi}{u_{30}} \quad (28)$$

Stability of $E_6 : \bar{N}_1 = -\left(e_1 + \frac{a_{13}e_3}{a_{11}}\right),\ \bar{N}_2 = 0,\ \bar{N}_3 = -e_3$

In this case, we have

$$\begin{vmatrix} (d_1 + a_{13}e_3) - \lambda & -\frac{a_{12}}{a_{11}}(d_1 + a_{13}e_3) & -\frac{a_{13}}{a_{11}}(d_1 + a_{13}e_3) \\ 0 & (g_2 - a_{23}e_3) - \lambda & 0 \\ 0 & 0 & d_3 - \lambda \end{vmatrix} = 0 \quad (29)$$

The characteristic roots are $d_1 + a_{13}e_3,\ g_2 - a_{23}e_3$ and d_3. Since two of these three roots are positive, hence, the state is unstable. The equations (6) yield the solutions,

$$u_1 = (u_{10} - \chi_1 - \chi_2)\, e^{(d_1 + a_{13}e_3)t} + \chi_1 e^{(g_2 - a_{23}e_3)t} + \chi_2 e^{d_3 t}; u_2 = u_{20} e^{(g_2 - a_{23}e_3)t}; u_3 = u_{30} e^{d_3 t} \tag{30}$$

where

$$\chi_1 = \frac{a_{13}(d_1 + a_{13}e_3)u_{20}}{a_{11}\left[(d_1 + a_{13}e_3 + a_{23}e_3) - g_2\right]} \text{ and } \chi_2 = \frac{a_{13}(d_1 + a_{13}e_3)u_{30}}{a_{11}\left[(d_1 + a_{13}e_3) - d_3\right]} \tag{31}$$

with

$$(d_1 + a_{13}e_3 + a_{23}e_3) \neq g_2 \text{ and } (d_1 + a_{13}e_3) \neq d_3 \tag{32}$$

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_1 = (u_{10} - \chi_1 - \chi_2)\left(\frac{u_2}{u_{20}}\right)^{\frac{d_1 + a_{13}e_3}{d_2 - a_{23}e_3}} + \frac{\chi_1 u_2}{u_{20}} + \chi_2\left(\frac{u_2}{u_{20}}\right)^{\frac{d_3}{d_2 - a_{23}e_3}}; \left(\frac{u_2}{u_{20}}\right)^{d_3} = \left(\frac{u_3}{u_{30}}\right)^{d_2 - a_{23}e_3} \tag{33}$$

Stability of $E_7 : \bar{N}_1 = \dfrac{a_{12}k_2}{a_{11}} - e_1,\ \bar{N}_2 = k_2,\ \bar{N}_3 = 0$

In this state, we get

$$\begin{vmatrix} (d_1 - a_{12}k_2) - \lambda & \frac{a_{12}}{a_{11}}(a_{12}k_2 - d_1) & \frac{a_{13}}{a_{11}}(a_{12}k_2 - d_1) \\ 0 & -g_2 - \lambda & a_{23}k_2 \\ 0 & 0 & -d_3 - \lambda \end{vmatrix} = 0 \tag{34}$$

The characteristic roots are $d_1 - a_{12}k_2, -g_2$ and $-d_3$.

Case I: When $d_1 < a_{12}k_2$

In this case all the three roots are negative, hence, the state is stable. The equations (6) yield the solutions,

$$u_1 = (u_{10} - \delta_1 - \delta_2)\, e^{-(d_1 + a_{12}k_2)t} + \delta_1 e^{-g_2 t} + \delta_2 e^{-d_3 t}; u_2 = (u_{20} - \delta)e^{-g_2 t} + \delta e^{-d_3 t}; u_3 = u_{30} e^{-d_3 t} \tag{35}$$

where

$$\delta_1 = \frac{a_{12}(d_1 - a_{12}k_2)(u_{20} - \delta)}{a_{11}(d_1 + g_2 - a_{12}k_2)};\ \delta_2 = \frac{(a_{12}\delta + a_{13}u_{30})(d_1 - a_{12}k_2)}{a_{11}(d_1 + d_3 - a_{12}k_2)};\quad \delta = \frac{a_{23}k_2 u_{30}}{g_2 - d_3} \tag{36}$$

with

$$d_1 + g_2 \neq a_{12}k_2;\ d_1 + d_3 \neq a_{12}k_2;\ g_2 \neq d_3 \tag{37}$$

It can be noticed that $u_1 \to 0, u_2 \to 0$ and $u_3 \to 0$ as $t \to \infty$

Case II: When $d_1 = a_{12}k_2$

In this case the state is neutrally stable and the solution curves of (6) are given by

$$u_1 = u_{10}; u_2 = (u_{20} - \delta)e^{-g_2 t} + \delta e^{-d_3 t}; u_3 = u_{30} e^{-d_3 t} \tag{38}$$

Case III: When $d_1 > a_{12}k_2$

In this case the state is unstable and the solution curves of (6) are given by

$$u_1 = (u_{10} - \delta_1 - \delta_2)\, e^{(d_1 + a_{12}k_2)t} + \delta_1 e^{-g_2 t} + \delta_2 e^{-d_3 t}; u_2 = (u_{20} - \delta)e^{-g_2 t} + \delta e^{-d_3 t}; u_3 = u_{30} e^{-d_3 t} \tag{39}$$

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_1 = (u_{10} - \delta_1 - \delta_2)\left(\frac{u_3}{u_{30}}\right)^{\frac{a_{12}k_2 - d_1}{d_3}} + \delta_1\left(\frac{u_3}{u_{30}}\right)^{\frac{g_2}{a_3}} + \frac{\delta_2 u_3}{u_{30}}; u_2 = (u_{20} - \delta)\left(\frac{u_3}{u_{30}}\right)^{\frac{g_2}{a_3}} + \frac{\delta u_3}{u_{30}} \tag{40}$$

Stability of $E_8(\bar{N}_1, \bar{N}_2, \bar{N}_3)$

In this case, we have

$$\begin{vmatrix} \beta_1-\lambda & \frac{a_{12}}{a_{22}}(g_2-a_{23}e_3) & \frac{a_{13}}{a_{22}}(g_2-a_{23}e_3) \\ 0 & (a_{23}e_3-g_2)-\lambda & \frac{a_{23}}{a_{22}}(g_2-a_{23}e_3) \\ 0 & 0 & d_3-\lambda \end{vmatrix}=0 \tag{41}$$

Where

$$\beta_1=d_1+a_{13}e_3+\frac{a_{12}a_{23}e_3}{a_{22}}-a_{12}k_2 \tag{42}$$

The characteristic roots are $\beta_1,\ a_{23}e_3-g_2$ and d_3. Since one of these three roots is positive, hence, the state is unstable. The equations (6) yield the solutions,

$$u_1=(u_{10}-\gamma_1-\gamma_2)\,e^{\beta_1 t}+\gamma_1 e^{(a_{23}e_3-g_2)t}+\gamma_2 e^{d_3 t};\ u_2=(u_{20}-\gamma)e^{(a_{23}e_3-g_2)t}+\gamma e^{d_3 t};\ u_3=u_{30}e^{d_3 t} \tag{43}$$

where

$$\gamma_1=\frac{a_{12}(g_2-a_{23}e_3)(\gamma-u_{20})}{a_{22}(g_2+\beta_1-a_{23}e_3)};\ \gamma_2=\frac{(a_{12}\gamma+a_{13}u_{30})(g_2-a_{23}e_3)}{a_{22}(d_3-\beta_1)};\ \gamma=\frac{a_{23}(g_2-a_{23}e_3)u_{30}}{a_{22}(d_3+g_2-a_{23}e_3)} \tag{44}$$

with

$$g_2+\beta_1\neq a_{23}e_3;\ d_3\neq\beta_1 \text{ and } g_2+d_3\neq a_{23}e_3 \tag{45}$$

The trajectories in the u_1-u_2 and u_2-u_3 planes are given by

$$u_1=(u_{10}-\gamma_1-\gamma_2)\left(\frac{u_3}{u_{30}}\right)^{\frac{\beta_1}{d_3}}+\gamma_1\left(\frac{u_3}{u_{30}}\right)^{\frac{a_{23}e_3-g_2}{d_3}}+\frac{\gamma_2 u_3}{u_{30}};\ u_2=(u_{20}-\gamma)\left(\frac{u_3}{u_{30}}\right)^{\frac{a_{23}e_3-g_2}{d_3}}+\frac{\gamma u_3}{u_{30}} \tag{46}$$

Liapunov's Function for Global Stability

We discussed the local stability of all eight equilibrium states. From which only two states E_3 and E_7 are stable and rest of them are unstable. We now examine the global stability of dynamical system (1), (2) and (3) at these states by suitable Liapunov's functions.

Theorem 1: The equilibrium state $E_3(0, k_2, 0)$ is globally asymptotically stable.

Proof: Let us consider the following Liapunov's function

$$L(N_2)=N_2-\bar{N}_2-\bar{N}_2\ln\left(\frac{N_2}{\bar{N}_2}\right)$$

Now, the time derivative of L, along with solution of (2) can be written as,

$$\frac{dL}{dt}=\left(\frac{N_2-\bar{N}_2}{N_2}\right)\frac{dN_2}{dt}$$

$$\frac{dL}{dt}=(N_2-\bar{N}_2)(g_2-a_{22}N_2)$$

$$\frac{dL}{dt}=(N_2-\bar{N}_2)(a_{22}\bar{N}_2-a_{22}N_2)$$

$$\frac{dL}{dt} = -a_{22}\left(N_2 - \bar{N}_2\right)^2 < 0$$

Hence, the steady state is globally asymptotically stable.

Theorem 2: The equilibrium state $E_7\left(\bar{N}_1, \bar{N}_2, 0\right)$ is globally asymptotically stable.

Proof: Let us consider the following Liapunov's function

$$L\left(N_1, N_2\right) = N_1 - \bar{N}_1 - \bar{N}_1 \ln\left(\frac{N_1}{\bar{N}_1}\right) + l_1\left[N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right)\right]$$

Where, l_1 is suitable constant to be determined in the subsequent steps.

Now, the time derivative of L, along with solutions of the equations (1) and (2) can be written as,

$$\frac{dL}{dt} = \left(\frac{N_1 - \bar{N}_1}{N_1}\right)\frac{dN_1}{dt} + l_1\left(\frac{N_2 - \bar{N}_2}{N_2}\right)\frac{dN_2}{dt}$$

$$\frac{dL}{dt} = \left(N_1 - \bar{N}_1\right)\left(-d_1 - a_{11}N_1 + a_{12}N_2\right) + l_1\left(N_2 - \bar{N}_2\right)\left(g_2 - a_{22}N_2\right)$$

$$\frac{dL}{dt} = \left(N_1 - \bar{N}_1\right)\left(a_{11}\bar{N}_1 - a_{12}\bar{N}_2 - a_{11}N_1 + a_{12}N_2\right) + l_1\left(N_2 - \bar{N}_2\right)\left(a_{22}\bar{N}_2 - a_{22}N_2\right)$$

$$\frac{dL}{dt} = -a_{11}\left(N_1 - \bar{N}_1\right)^2 + a_{12}\left(N_1 - \bar{N}_1\right)\left(N_2 - \bar{N}_2\right) + l_1\left[-a_{22}\left(N_2 - \bar{N}_2\right)^2\right]$$

$$\frac{dL}{dt} = \left[\left\{\sqrt{a_{11}}\left(N_1 - \bar{N}_1\right) - \sqrt{a_{22}l_1}\left(N_2 - \bar{N}_2\right)\right\}^2 + \left(2\sqrt{a_{11}a_{22}l_1} - a_{12}\right)\left(N_1 - \bar{N}_1\right)\left(N_2 - \bar{N}_2\right)\right]$$

Choosing $l_1 = \frac{a_{12}^2}{4a_{11}a_{22}} > 0$ and with some algebraic manipulation, we get

$$\frac{dL}{dt} = -\sqrt{a_{11}}\left[\left(N_1 - \bar{N}_1\right) - \frac{a_{12}}{a_{11}}\left(N_2 - \bar{N}_2\right)\right]^2 < 0$$

Hence, the steady state is globally asymptotically stable.

Conclusion

The present paper deals with an investigation on the stability of a syn eco-system consisting of two hosts and one commensal with mortality rate for the first and third species. In this paper we established all possible equilibrium states. It is conclude that, in all eight equilibrium states, only the two states E_3 and E_7 are stable. Further the global stability is established with the help of suitable Liapunov's function.

REFERENCES

Acharyulu KVLN and Charyulu NChPR (2011). An Ammensal-Prey with three species Ecosystem. *International Journal of Computational Cognition* **9** 30-39.

Colinvaux AP (1986). *Ecology* (John Wiley, New York, USA).

Kapur JN (1985). *Mathematical Modeling in Biology and Medicine* (Affiliated East West Press, New Delhi, India).

Kapur JN (1985). *Mathematical Modelling,* (Wiley Easter).

Kumar NP (2010). Some Mathematical Models of Ecological Commensalism. Acharya Nagarjuna University, Ph.D Thesis.

Kushing JM (1977). *Integro-Differential Equations and Delay Models in Population Dynamics.* Lecture Notes in Bio-Mathematics (Springer Verlag, New York, USA) **20**.

Lotka AJ (1925). *Elements of Physical Biology* (Williams and Wilking, Baltimore, USA).

Meyer WJ (1985). *Concepts of Mathematical Modeling* (Mc.Graw Hill, New York, USA).
Narayan KL and Charyulu NChPR (2007). A Prey-Predator Model with Cover for Prey and Alternate Food for the Predator and Time Delay. *International Journal of Scientific Computing* **1** 7-14.
Prasad BH (2014). On the Stability of a Three Species Syn-Eco-System with Mortality Rate for the Second Species. *International Journal of Social Science & Interdisciplinary Research* **3** 35-45.
Prasad BH (2014). The Stability Analysis of a Three Species Syn-Eco-System with Mortality Rates. *Contemporary Mathematics and Statistics* **2** 76-89.
Prasad BH (2015). A Mathematical Study on Syn-Eco-System Consisting of Two Hosts and One Commensal with Mortality Rate for the Third Species. *International Journal of Physics and Mathematical Sciences* **5** 78-86.
Prasad BH (2015). A Study on Discrete Model of a Typical Three Species Syn-Ecology with Limited Resources. *International Journal of Animal Biology* **1** 69-73.
Prasad BH (2016). A Study on Discrete Model of Three Species Syn-Eco-System with Unlimited Resources for the First species. *International Journal of Mathematical Sciences, Technology and Humanities* **6** 01-21.
Ravindra Reddy R (2008). A Study on Mathematical Models of Ecological Mutualism between Two Interacting Species. Osmania University Ph.D Thesis.
Reddy RA, Charyulu NChPR and Gandhi BK (2007). A Stability Analysis of Two Competetive Interacting Species with Harvesting of Both the Species at a Constant Rate. *International Journal of Scientific Computing* **1** 57-68.
Sharma BBR and Charyulu NChPR (2008). Stability Analysis of Two Species Competitive Eco-system. *International Journal of Logic Based Intelligent Systems* **2** 79-86.
Srinivas NC (1991). *Some Mathematical Aspects of Modeling in Bio-medical Sciences*. Kakatiya University, Ph.D Thesis.
Volterra V (1931). *Leconssen La Theorie Mathematique De La Leitte Pou Lavie*. (Gauthier-Villars, Paris, France).