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Research Article

INFLUENCE OF VERSES OF *LĪLĀVATĪ* WRITTEN BY *BHĀSKARA-II* IN PRESENT SCHOOL MATHEMATICS

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ABSTRACT
Līlāvatī[1] may be conceptually expressed as *'Līlā'* means play with mathematical themes and *'vatī'* means terminating i.e. giving some results. *Līlāvatī* may be subdivided into a) Arithmetic[2]; b) Algebra[3]; c) Trigonometry & Geometry and d) Discrete Mathematics. Here we are considering those verses, from *Bhāskarācārya's* vast propositions, which can assume algebraic notations and used in Junior school mathematics. For these we have taken a few fragments of mathematical operations.

Keywords: *Līlāvatī, BHĀSKARA-II*

INTRODUCTION
Bhāskarācārya's[4] work in Algebra, Arithmetic, Geometry, Discrete Mathematics made him to exist in the peak of his fame and immortality. His illustrious mathematical works within *Līlāvatī* and *Bījagaṇitam* are considered to be the incomparable and memorial to his profound intelligence. Its translations in several languages throughout the world bear his testimony to its eminence. He is called *'Gaṇakacakracūdāmaṇi'*[5] meaning thereby: *'top of the all-round arithmeticians'.*
He wrote about his year of birth as:
अथप्रश्नाध्यायः / श्लोक–५८ (इदानींसिद्धान्तग्रन्थनकालमाह)
रसगुणपूर्णमही१०३६समशकनृपसमयेऽभवन्ममोत्पत्तिः।
रसगुण३६वर्षेणमयासिद्धांतशिरोमणीःरचितः॥

Transcription
Atha praśnādhyāyaḥ / Śloka-58 (Idānīṁ siddhāntagranthanakālamāha)
Rasaguṇapūrṇamahī1036samaśakanṛpasamaye'bhavanmamotpattiḥ।
Rasaguṇa36varṣeṇa mayā siddhāṁtaśiromaṇīḥ racitaḥ॥
In Chapter of Problems & Questionnaire in the fourth part of Siddhānta Śiromaniḥ i.e. in Golādhyāyaḥ in Stanza – 58 (Year of his birth & of Siddhānta Śiromaniḥ):-
'I was born in Śaka 1036 (1114 AD or CE) and I wrote Siddhānta Śiromaniḥ when I was 36 years old' i.e. in Saka 1072 (1150 CE).
We find about Pāṭigaṇitam at end of the first stanza of Līlāvatī:
पाटींसतङ्गणितस्यवच्मिचतुरप्रीतिप्रदांप्रस्फुटां।
सङ्क्षिप्ताक्षरकोमलामलपदैर्लालित्यलीलावतीम्॥

Transcription
Patīṁ sadgaṇitasya vacmi caturaprītipridāṁ pradāṁ prasphuṭāṁ |
Saṅkṣistākṣarakomalāmalapadairlālityalīlāvatim || 1

English Version
I propound this easy process of computation with soft and correct as well as concise words, delighting the elegance and the learned with great satisfaction.
In following verse, we find *Bhāskarācārya*'s mastership as *Ācārya* (Preceptor):
अष्टौव्याकरणानिषट्चभिषजांव्याचष्टताःसंहिताः
षट्तर्कान्गणितानिपञ्चचतुरोवेदानधीतेस्मयः।

रत्नानांत्रितयंद्वयञ्चबुबुधेमीमांसयोरन्तरे
सद्ब्रह्मैकमगाधबोधमहिमासोऽस्याकविभास्करः॥२७९

Transcription

Aṣṭau vyākaraṇāni ṣaṭ ca bhiṣajā vyācaṣṭa tāḥ saṁhitāḥ
Ṣaṭ tarkān gaṇitāni pañca caturo vedānadhīte sma yaḥ|
Ratnānāṁ tritayaṁ dvayañca bubudhe mīmāṁsayorantare
Sadbrahmaukamagādhabodhamahimā so'syā kavibhāskaraḥ||[6] 278

Bhāskarācārya, the great poet and author of *Sidhānta Śiromaniḥ*[7], had mastered eight volumes on Grammar[8], six on Medicine& Medical Sciences[9], six on Philosophical Systems[10], five on Mathematics[11], four Vedas[12], a triad of three *ratnas*[13] and two (fore & past) *Mīmāṁsās*[14]. He understood that the Lord (Supreme) cannot be fathomed.

He mastered not only mathematics but also in many branches of philosophy & science.

Bhāskara-II was head of the astronomical observatory at Ujjain, the leading mathematical centre of ancient India. His predecessors in this post were noted Indian mathematicians Varāhamihira[15] and Brahmagupta[16]. *Siddhānta Śiromaniḥ* is a mammoth work containing about 1450 verses dividing it into four different concepts as: *Līlāvatī, Bījagaṇitam, Gaṇitādhyāyaḥ*[17] *(Grahagaṇitam)* and *Golādhyāyaḥ*[18] whereas each part has a separate identity, therefore, may be considered as separate book. The numbers of verses in each part are: In Līlāvatī – 278 verses, in Bījagaṇitam – 213 verses, in Gaṇitādhyāyaḥ – 451 verses and in Golādhyāyaḥ – 501 verses. Most beautiful characteristic of *Siddhānta Śiromaniḥ* is that it expressed most simple methods of calculations from Arithmetic to Astronomy.

Līlāvatī had been divided into 13 chapters covering many branches of mathematics such as: arithmetic, algebra, geometry, and a little trigonometry and mensuration. More specifically the contents are:

- Definitions.
- Properties of zero (including division, and rules of operations with zero).
- Further extensive numerical work, including use of negative numbers and surds.
- Estimation of π.
- Arithmetical terms, methods of multiplication, and squaring.
- Inverse rule of three, and rules of 3, 5, 7, 9, and 11.
- Problems involving interest and interest computation.
- Arithmetical and geometrical progressions.
- Plane (geometry).
- Solid geometry.
- Permutations and combinations.
- Indeterminate equations (Kuttaka), integer solutions (first and second order). His contributions to this topic are particularly important, since the rules he gives are (in effect) the same as those given by the renaissance European mathematicians of the 17th century, yet his work was of the 12th century. Bhāskara-II's method of solving was an improvement of the methods found in the work of Āryabhaṭṭa and subsequent mathematicians.

In nineteenth century a great German Mathematician *Karl Theodor Wilhelm Weierstrass*[19] said:

". . . es ist wahr, ein Mathematiker, der nicht etwas Poet ist, wird nimmer ein vollkommener Mathematiker sein".

". . . it is true that a mathematician, who is also not something of a poet, can never be a complete mathematician"[20].

MATERIALS AND METHODS

Methods

We have taken those Sanskrit text from *Līlāvatī* which are appropriate to the school level mathematics.

Rules of Multiplication

गुणनेकरणसूत्रंसार्द्धवृत्तद्वयम (द्वितीयोऽध्यायः / द्वितीयःपरिच्छेदः):-
गुण्यान्त्यमङ्कंगुणकेनहन्यादुत्सारितेनैवमुपान्तिमादीन्।

गुण्यस्त्वधोऽधोगुणखण्डतुल्यस्तैःखण्डकैःसङ्गुणितोयुतोवा॥१४
भक्तोगुणोःशुध्यतियेनतेनलब्ध्याचगुण्योगुणितःफलंवा।
द्विधाभवेद्रूपरिभागएवंस्थानैःपृथग्वागुणितःसमेतः।
इष्टोनयुक्तेनगुणेननिघ्नोऽभीष्टघ्नगुण्यान्वितवर्जितोवा॥१५

Transcription:

Guṇane karaṇasutraṁ sārddhavṛttadvay̤am (Dvitīy̤o'dyāy̤aḥ / Dvitīy̤aḥ pricchedaḥ):-
Guṇyāntyamaṅkaṁ guṇakena hanyādutsāritenaivamupāntimādīn|
Guṇyastvadho'dhoguṇakhaṇḍatulyastaiḥ khaṇḍakaiḥ saṅguṇito yuto vā|| 14
Bhaktoguṇoḥ śudhyati yena tena labdhyā ca guṇyo guṇitaḥ phalaṁ vā|
Dvidhā bhavedrūparibhāga evaṁ sthānaiḥ pṛthagvā guṇitaḥ smetaḥ|
Iṣṭonayuktena guṇena nighno'bhīṣṭaghnaguṇyānvitavarjito vā|| 15

I am describing the 'Rules of Multiplication' in two-and-half stanza:

Rule-1: Digitalised Method – Digit in unit-place of the multiplicand to be multiplied by multiplier then the tenth digit and so on repeating up to the last digit on the extreme left. Symbolically, $a = 10a'' + a'$ where *'a''* is the unit digit & *'a''* is the tenth digit of the multiplicand then $ab = 10a''b + a'b$ where b is the multiplier.

Rule-2: Split up Method – Split the multiplier into two convenient parts then multiply the multiplicand by each of the two parts and add the result. –. Symbolically it may be expressed as $a(b + c) = ab + ac$ where *'a'* is multiplicand and *'b + c'* is split up of multiplier.

Rule-3: Factorisation Method – If multiplier is factorable number or composite number then split it into factors. Multiply the multiplicand by one factor then multiply the product by second factor and so on. Symbolically, $(ab)c$ where *'a'* is multiplicand and *'b'* & *'c'* are factors of multiplier.

Rule-4: Placement Method – Multiply the multiplicand by each digit of the multiplier and place the result according placement of multiplier (i.e. result with unit digit from unit place & result with tenth digit from place). Then add. Symbolically, $a(10b + c) = 10ab + ac$ where *'a'* is multiplicand and *'b'* is the digit at tenth place& *'c'* is the digit of unit place of the multiplier.

Rule-5: Adding & Subtracting Method:

a) Add any assumed number to the multiplier to make it easy for multiplication. Multiply the multiplicand by added number. Then multiply the multiplicand by assumed number. Now subtract the results. Symbolically, $ab = a(b + c) – ac = ab + ac – ac$ where *'a'* is multiplicand, *'b'* is multiplier and *'c'* is assumed convenient number.

b) Subtract any assumed number to the multiplier to make it easy for multiplication. Multiply the multiplicand by subtracted number. Then, multiply the multiplicand by assumed number. Now add the results. Symbolically, $ab = a(b – c) + ac = ab – ac + ac$ where *'a'* is multiplicand, *'b'* is multiplier and *'c'* is assumed convenient number.

अत्रोद्देशकः।
बालेकुरङ्नलोलनयनेलीलावतिप्रोच्यतांपञ्चत्येकमितादिवकरगुणअङ्काकतिस्युर्यदि।
रुपस्थानविभागखण्डगुणनेकस्थासिकस्थाणिनिछिन्नास्तेनगुणेनतेचगणिताजाताःकतिस्युर्वद॥१६

Transcription

Atroddeśakaḥ |
Bāle kuraṅnalolanay̤ane līlāvati procyatāṁ
Pañcatyekamitādivakaraguṇa aṅkā katisyuryadi |
Rupasthānavibhāgakhaṇḍaguṇane kasthāṇini
Chinnāstena guṇena te ca gaṇitā jātāḥ kati syurvada || 16

English Version

Example:

Oh! Beautiful and dear Līlāvatī with eyes like fawn, tell me the number resulting from one hundred and thirty-five taken into twelve. If you be skilled in multiplication by whole or by parts whether by subdivision of form or separation of digits. Tell me, auspicious woman, the quotient of the product divided by the same multiplier.

न्यासः।गुण्यः।१३५।गुणकः।१२।

गुण्यान्त्यभङ्कंगुणकेनहन्यादितिकृतेजातम्।१६२०।

Transcription

Nyāsaḥ | Guṇyaḥ | 134 | Guṇakaḥ | 12 |

Guṇyāntyabhaṅkaṁ guṇakena hanyāditi kṛte jātam | 1620 |

Solution:

Multiplicand 135. Multiplicator 12.

Product, multiplying the digits of the multiplicand successively by the multiplicator = 1620.

135 × 12 = 1620				
5 × 12			6	0
3 × 12		3	6	
1 × 12	1	2		
Addition	1	6	2	0

अथवागुणरुपविभागेखण्डेकृते।८।४।आभ्यांपृथग्गुण्येगुणितेयुतेचजातंतदेव।१६२०।

Transcription

Athavā guṇarupavibhāge khaṇḍe kṛte | 8 | 4 | Ābhyāṁ pṛthagguṇye guṇite yute jātaṁ tadeva | 1620 |

Solution:

Subdividing the multiplicator into two parts as: 8 + 4 = 12. Severally multiplying the multiplicand by them: 135 × 8 = 1080; 135 × 4 = 540. Add the product together = 1080 + 540 = 1620.

135 × 12 = 1620				
135 × 8	1	0	8	0
135 × 4		5	4	0
Addition	1	6	2	0

अथवागुणकस्त्रिभिर्भक्तोलव्धं।४।एभिस्त्रिभि-(३) श्चगुणेगुणितेजातंतदेव।१६२०।

Transcription

Athavā guṇakastribhirbhaktolavdhaṁ | 4 | Ebhistribhi-(3) śca guṇe guṇite jātaṁ tadeva |1620|

Solution

Or, multiplicator is divided by 3 then quotient is 4 i.e. 12 ÷ 3 = 4. Successively multiply by 4 and 3, the last product = 1620. [135 × 4 = 540 × 3 = 1620]

135 × 12 = 1620				
First 135 × 4		5	4	0
Then 540 × 3	1	6	2	0

अथवास्थानविभागेखण्डे।१।२।आभ्यांपृथग्गुण्येगुणितेयथास्थानयुतेचजातंतदेव।१६२०।

Transcription

Athavā sthānavibhāge khaṇḍe | 1 | 2 | Ābhyāṁ pṛthagguṇye guṇite yathāsthānayute ca jātaṁ tadeva |1620|

Solution:

Or, considering digits of multiplicator as parts viz. 1 and 2. Multiply the multiplicand by them severally. The products added together according to the places of figures, then result is 1620.

135 × 12 = 1620				
135 × 2		2	7	0
135 × 1	1	3	5	
Addition	1	6	2	0

अथवाद्व्युनेनगुणेन (१०)द्वाभ्याञ्च (२) पृथग्गुण्येगुणितेयुतेचजातंतदेव।१६२०।

Transcription
Athavā dvyunena guṇena (10) Dvābhuāñca (2) Pṛthagguṇye guṇite yute ca jātaṁ tadeva | 1620 |
Solution
Or, multiplicand is multiplied by multiplicator less 2 = 12 – 2 = 10 and then result is added to twice the multiplicand, the net result is 1620. [(135 × 10) + (135 × 2) = 1350 + 270 = 1620]

135 × 12 = 1620				
135 × 10	1	3	5	0
135 × 2		2	7	0
Addition	1	6	2	0

अथवाष्टयुतेनगुणेन (२०) गुण्येगुणितेऽष्ट-(८) गुणितगुण्यहीनेचजातंतदेव।१६२०।
Transcription
Athavāṣṭayutena guṇena (20) Guṇye guṇite'ṣṭa-(8) Guṇitaguṇyahīne ca jātaṁ tadeva | 1620 |
Solution
Or, the multiplicand is multiplied by the multiplicator increased by 8 viz. 20. Then 8 times of the multiplicator being subtracted result derived = 1620 [(135 × 20) – (135 × 8) = 2700 – 1080 = 1620.

135 × 12 = 1620				
135 × 20	2	7	0	0
(–) 135 × 8	1	0	8	0
After Subtraction	1	6	2	0

Rules of Squaring
वर्गेकरणसूत्रंवृत्तद्वयम्।
समद्विधातःकृतिरुच्यतेऽथस्थाप्योऽन्त्यवर्गोद्विगुणान्त्यनिघ्ना।
स्वस्वोपरिष्ठाच्चतथापरेऽङ्कास्त्यक्त्वान्त्यमुत्सार्यपुनश्चराशिम्॥१८
खण्डद्वयस्याभिहतिर्द्विनिघ्नीतत्खण्डवर्गैक्ययुताकृतिर्वा।
इष्टोनयुग्राशिवधःकृतिःस्यादिष्टस्यवर्गेणसमन्वितोवा॥१९
Transcription
Varge karaṇsūtraṁ vṛttadvayam |
Samadvighātaḥ kṛtirucyate'tha sthāpyo'ntyavargodviguṇāntyanighnā |
Svasvopariṣṭācca tathā pare'ṅkāstyaktvāntyamutsārya punaśca rāśim || 18
Khaṇḍadvayasyābhihatirdvinighnī tat khaṇḍavargaikyayutā kṛtirvā |
Iṣṭonayugrāśivadhaḥ kṛtiḥ syādiṣṭasya vargeṇa samanvito vā || 19
Rule for the square of a quantity: two stanzas:-
Rule-1: Definition of square number – The multiplication of a numbers twice is the square of that number or the product of a number with itself is called its square.
Rule-2: Procedure of squaring a number – Square the last number i.e. extreme left-hand / extreme right-hand digit and the rest of the digits doubled and multiplied by the last; then repeating for next. Symbolically, operating from left may be expressed as $(a+10b+100c)^2 = a^2 + 2a(b + 10c) + b^2 + 2bc + c^2$ where $(b + 10c)$ represents number with digits of tenth & hundredth places and *'a', 'b', 'c'* are digits of unit, tenth and hundredth places respectively. Now, operating from right or $(a+10b+100c)^2 = c^2 + 2c(b + 10a) + b^2 + 2ab + a^2$; similarly, $(10b + a)$ represents number with digits of tenth & unit places digits and where *'a', 'b', 'c'* are digits of unit, tenth and hundredth places respectively. Considering *a*, *b*, *c* as different algebraic quantities the expression may be taken as: $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$ which is algebraic formula of square of sum of three different quantities.
अत्रोद्देशकः।
सखेनवनाश्चचतुर्दशानांब्रूहित्रिहीनस्यशतत्रयस्य।
पञ्चोत्तरस्याप्ययुतस्यवर्गंजानासिचेद्वर्गविधानमार्गम्॥२०

Transcription:
Atroddeśakaḥ |
Sakhe navanāñca caturdaśānāṁ vrūhi trihīnasya śatatrayasya |
Pañcottarasyāpyayutasya vargaṁ jānāsi cedvargavidhānamārgam || 20
Now, example:
Oh! Friend (woman) do you know the method of computing squares of nine, fourteen, three less than three-hundred and five more than ten thousand.
न्यासः।९।१४।२९७।१०००५।
एषांयथोक्तकरणेनजातावर्गाः।८१।१९६।८८२०९।१००१०००२५।
Transcription
Nyāsaḥ | 9 | 14 | 297 | 10005 |
Exam yathoktakaraṇena jātāvargāḥ | 81 | 196 | 88209 | 10010025 |
English version
We are considering to find squares of 9, 14, 297, 10005.
Proceeding as per desired method, squares may be derived as | 81 | 196 | 88209 | 10010025 |
अथवानवानांखण्डे।४।५।अनयोराहति—।२०।द्विर्निघ्नी।४०।तत्खण्डवर्गैक्येन।४१।युताजातासैवकृतिः।८१।
Transcription
Athavā nabānāṁ khaṇḍe | 4 | 5 | Anayorāhati — | 20 | Dvirnighnī | 40 | Tatkhaṇḍavargaikyena | 41 | Yutā jātā saiva kṛtiḥ | 81 |
Solution:
Split 9 as 4 + 5. Product of parts is 20. Double their product to get 40. Sum of the squares of the parts = $4^2 + 5^2 = 41$. Adding we get 40 + 41 = 81 [Algebraically, $(4 + 5)^2 = 2 \times 4 \times 5 + 4^2 + 5^2$ that it tallies with the formula $(a + b)^2 = a^2 + 2ab + b^2$].
अथवाचतुर्दशानांखण्डे।६।८।अनयोराहति—
।४८।द्विर्निघ्नी।९६।तत्खण्डवर्गौ।३६।६४।अनयोरैक्येन।१००।युातायातासैवकृतिः।१९६।
Transcription
Athavā caturdaśānāṁ khaṇḍe | 6 | 8 | Anayorāhati | 48 | Dvirnighnī | 96 | Tatkhaṇḍavargai | 36 | 64 | Anayoraikyena | 100 | Yuta jātā saiva kṛtiḥ | 196 |
Solution
Now split 14 as 6 + 8. Product of the parts is 48. Twice the product is 96. Squares of the two parts are 36 & 64. Sum of the squares is 100. Adding we get 96 + 100 = 196 [Algebraically, $(6 + 8)^2 = 2 \times 6 \times 8 + 6^2 + 8^2$; that it tallies with the formula $(a + b)^2 = a^2 + 2ab + b^2$].
अथवाखण्डे।४।१०।तथापिसैवकृतिः।१८६।
Transcription:
Athavā khaṇḍe | 4 | 10 | Tathāpi saiva kṛtiḥ | 196|
Solution
Splitting 14 as 4 + 10 we can proceed same way above to get its square as 196.
अथवाराशि।२९७।अय़ंत्रिभिरुनितःपृथग्युतश्च।२९४।३००।अनयोर्घातः।८८२००।त्रिवर्ग—।९।युतोजातोवर्गःसएव।८८२०९।
Transcription
Athavā rāśi | 297 | Ayaṁ trivirunitaḥ pṛthagyutaśca | 294 | 300 | Anayorghātaḥ | 88200 | Trivarga — | 9 | Yuto jātovargaḥ sa eba | 88209 |
Solution
Now, consider 297. Diminishing 3 from it we get 294 and increasing by 3 get 300. Product of these two = 294 × 300 = 88200. Adding square of 3 i.e. 9 we get 88209 which is square of 297. [Algebraically, $(a - b)(a + b) + b^2 = a^2$; where a = 297 & b = 3]
It may also be expressed by the position digital formula: $(100a + 10b + c)^2 = (100a)^2 + 2 \times 100a \times 10b + 2 \times 100a \times c + (10b)^2 + 2 \times 10b \times c + c^2$; where *a, b, c* are digits of 100th, 10th, unit places respectively as $(297)^2 = (200)^2 + 2\times200\times90 + 2\times200\times7 + (90)^2 + 2\times90\times7 + 7^2 = 40000 + 36000 + 2800 + 8100 + 1260 + 49 = 88209$.

We may take: 297 = 290 + 7. Then $(290)^2$ = 841000, $(7)^2$ = 49, 2×290×7 = 4060. So, $(297)^2$ = 841000 + 49 + 4060 = 88209 which is congruence to the algebraic expression: $(a+b)^2 = a^2 + b^2 + 2ab$.
Example: -

Square of 297						**Square of 297**					
Operation from right	**Value**					**Operation from left**	**Value**				
7^2				4	9	2^2	4				
$7 \times 2 \times 29$		4	0	6		$2 \times 2 \times\times 97$	3	8	8		
9^2		8	1			9^2		8	1		
$9 \times 2 \times 2$	3	6				$9 \times 2 \times 7$		1	2	6	
2^2	4					7^2				4	9
	8	8	2	0	9		8	8	2	0	9

We take: 10005 = 10000 + 5. Then $(10005)^2 = (10000)^2 + 2 \times 10000 \times 5 + (5)^2$ = 100000000 + 100000 + 25 = 100100025 which is congruence to the algebraic expression: $(a + b)^2 = a^2 + 2ab + b^2$.

Rule-3: *Split up Method* – Split the given number into two convenient parts. Square the parts separately. Now with sum of the squares of the two parts add twice the product of the two parts to give the result. Symbolically, $(a + 10b)^2$ *or* $(10a + b)^2 = a^2 + b^2 + 2ab$where *'a', 'b'* are digits of unit place and tenth place respectively or vice-versa. Considering *a* and *b* to be two different algebraic quantity we can deduce well known formula: $(a + b)^2 = a^2 + 2ab + b^2$.
Rule-4: Adding & Subtracting Method – Add and subtract a suitable number with the given number whose square to be done. Then take the product of sum and difference; add square of the chosen number to get the result. Symbolically, $(a + b)(a – b) + b^2 = a^2$ where *'a'* is given number & *'b'* is suitably chosen number.
Rules of Cubing
घनेकरणसूत्रंवृत्तत्रय़म्।
समत्रिघातश्चघनःप्रदिष्टःस्थाप्योघनोन्त्यस्मततोऽन्त्यवर्गः।
आदित्रिनिघ्नस्ततआदिवर्गस्त्र्यन्त्याहतोऽथादिघनश्चसर्वे॥२३
स्थानान्तरत्वेनयुतोघनःस्यात्प्रकल्प्यतत्खण्डयुगंततोऽन्त्यम्।
एवंमुहुर्वर्गघनप्रसिद्धावाद्याङ्कतोवाविधिरेषकार्य्यः॥२४
Transcription
Ghane karaṇasūtraṁ vṛttatrayam |
Samatrighātaśca ghanaḥ pradiṣṭaḥ sthāpyoghanontyastma tato'ntyavargaḥ |
Āditrinighnastata ādivargastryantyāhato'thādighanaśca sarbe || 23
Sthānāntaratvena yuto ghanaḥ syāt prakalpya tat khaṇḍayugaṁ tato'ntyam |
Ebaṁ muhurvargaghanaprasiddhā vādyāṅkato vā bidhireṣa kāryyaḥ || 24
Rule-1: Definition – The repeated multiplication of a quantity thrice is a cube. Cube of a given number is its product with itself thrice.
Rule-2: Cube of two-digit number – The cube of the last (digit) [i.e. extreme left] is to be done; then square of the last digit multiplied by three times the first; square; afterwards square of the first taken into last and tripled and lastly cube of first; add together according to their places, make the cube. It may be expressed as: let the two-digit number is 10a + b where 'a' is digit of tenth place and 'b' is of unit place. Then, to find cube of the number find a^3 first, below this find $3a^2b$ and place it after shifting one place towards right, below this find $3ab^2$ and place it after shifting one place towards right again, below this find b^3 and again place it after shifting one place towards right. Add all this results to get cube. This process may be modified with starting from 'b' but then each time the shifting should be made to the left. Symbolic representation of cube of sum of two quantities is: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
खण्डाभ्यांवाहतोराशिस्त्रिघ्नःखण्डघनैक्ययुक्।
वर्गमूलघनस्वघ्नोवर्गराशेर्घनोभवेत्॥२६
Transcription

Khaṇḍābhyāṁ vā hato rāśistrighnaḥ khaṇḍaghanaikyayuk |
Vargamūlaghana svaghno vargarāśerghano bhavet || 26
Solution
Split the given number into two parts. Multiply its product by three times its sum and find the sum of the cubes of parts. Total of these is the required cube.
Cube of a given number is the square of the cube of square-root of the given number.
$(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
$\left\{\left(\sqrt{a^2}\right)^3\right\}^2 = (a^2)^3$
अत्रोद्देशकः।
नवघनंत्रिघनस्यघनंतथाकथयपञ्चघनस्यघनञ्चमे।
घनपदञ्चततोऽपिघनात्सखेयदिघनेऽस्तिघनाभवतोमतिः॥२७
Transcription
Atroddeśakaḥ |
Nabaghanaṁ trighanasya ghanaṁ tathā kataya pañcaghanasya ghanañca me |
Ghanapadañca tato'pi ghanāt sakhe yadi ghane'sti ghanā bhavato matiḥ ||27
English version
Oh! My friend (woman), tell me, the cube of nine, the cube of cube of three and the cube of cube of five as well as cube-roots of those cubes if thy knowledge is enough to compute cube.
न्यासः।९।२७।१२५।
जाताःक्रमेणघनाः।७२९।१९६८३।१९५३१२५।
Transcription
Nyāsaḥ | 9 | 27 | 125 |
Jātāḥ krameṇa ghanāḥ | 729 | 19683 | 1953125 |
English version
Consider finding of cubes of 9, 27, 125
Cubes of above numbers in order are: 729, 19683, 1953125.
अथवाराशिः।९।अस्यखण्डे।४।५।आभ्यांराशिर्हतः।१८०।त्रिनिघ्नञ्च।५४०।खण्डघणैक्येन।१८९।युतोजातोघनः।७२९।
Transcription
Athavā rāśiḥ | 9 | Asya khaṇḍe | 4 | 5 | Ābhyāṁ rāśirhataḥ | 180 | Trinighnañca | 540 | Khaṇḍaghaṇaikyena | 189 | Yuto jato ghanaḥ | 729 |
Solution
Consider '9' for cubing. Split 9 as 4 + 5. Multiply them with 9 = 4 × 5× 9 = 180. Then, multiply the result by 3 = 180 × 3 = 540. Sum of the cubes of the parts = $4^3 + 5^3$ = 64 + 125 = 189. Adding we get cube of 9 = 729 (= 540 + 189).
अथवाराशिः।२७।अस्यखण्डे।२०।७।आभ्यांहतस्त्रिघ्नश्च।११३४०।खण्डघनैक्येन।८३४३।युतोजातोघनः।१९६८३।
Transcription
Athavā rāśiḥ | 27 | Asya khaṇḍe | 20 | 7 | Ābhyāṁ hatastrighnañca | 11340 | Khaṇḍaghanaikyena | 8343 |Yuta jāto ghanaḥ | 19683 |
Solution
Consider '27' for cubing. Entire number being 27, make into two parts as 27 = 20 + 7. Number being multiplied successively with the parts and 3 we get = 27 × 20 × 7 × 3 = 11340. Sum of the cubes of the parts = $20^3 + 7^3$ = 8000 + 343 = 8343. Adding we get the cube of 27 = 11340 + 8343 = 19683.
अथवाराशिः।४।अस्यमूलम्।२।घनः।८।अयंस्वघ्नोजातश्चतुर्णांघनः।६४।
Transcription
Athavā rāśiḥ | 4 | Asya mūlam | 2 | Ghanaḥ | 8 | Ayaṁ svaghnojātañcaturṇā ghanaḥ | 64 |
Solution

Here proposed square number is 4. Its square-root is 2 then cubed 8 then squared equals to 64 which is cube of 4. (Algebraically used formula is $\left\{\left(\sqrt{a^2}\right)^3\right\}^2 = (a^2)^3$ where $a^2 = 4$).

अथवाराशिः।९।अस्यमूलम्।३।घनः।२७।अस्यवर्गोनवानांघनः।७२९।यएववर्गराशिघनःसएववर्गमूलघनवर्गः।

Transcription

Athavā rāśiḥ | 9 | Asya mūlam | 3 | Ghanaḥ | 27 | Asya vargonavānāṁ ghanaḥ | 729 | Ya eba vargarāśighanaḥ sa eba vargamūlaghanavargaḥ |

Solution:

Here proposed square number is 9. Its square-root is 3 then cubed 27 then squared equals to 729 which is cube of 9. This is the representation of algebraically formula is $\left\{\left(\sqrt{a^2}\right)^3\right\}^2 = (a^2)^3$ where $a^2 = 9$).

Example:

Cube of 27 (Shifting Right)						**Cube of 27 (Shifting Left)**					
Operation from Right	**Value**					**Operation from Left**	**Value**				
$(2)^3$		8				$(7)^3$			3	4	3
$3(2)^2 7$		8	4			$3(7)^2 2$		2	9	4	
$3(2)(7)^2$		2	9	4		$3(7)(2)^2$		8	4		
$(7)^3$			3	4	3	$(2)^3$		8			
$(27)^3$	1	9	6	8	3	$(27)^3$	1	9	6	8	3

Rule-3: Cube of more than two-digit number – If there are more than two digits, then make it into group of two digits then find the cube of the two digits at the extreme left and continue with the procedure above. This method recognize the algebraic identity $(a + b + c)^3 = (a + b)^3 + 3(a + b)^2c + 3(a + b)c^2 + c^3$.

Rule-4: Split up Method – Split the given number, to be cubed, in two parts. Find cubes of each parts. Multiply their product with three times of their sum. Add them and get the result. This method indicates algebraic identity $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$.

Example:

Cube of 125 (Shifting Right)								**Cube of 125 (Shifting Left)**							
Making Two Groups (12)5								**Making Two Groups (12)5**							
Operation from Right	**Value**							**Operation from Left**	**Value**						
$(12)^3$	1	7	2	8				$(5)^3$					1	2	5
$3(12)^2 5$		2	1	6	0			$3(5)^2 12$				9	0	0	
$3(12)(5)^2$				9	0	0		$3(5)(12)^2$		2	1	6	0		
$(5)^3$					1	2	5	$(12)^3$	1	7	2	8			
$(125)^3$	1	9	5	3	1	2	5	$(125)^3$	1	9	5	3	1	2	5

Example: 27 may be split up as 20 + 7

Cube of 27 (Shifting Right)						**Cube of 27 (Shifting Left)**					
Operation from Right	**Value**					**Operation from Left**	**Value**				
$(20)^3$		8	0	0	0	$(7)^3$			3	4	3
$(7)^3$			3	4	3	$(20)^3$		8	0	0	0
$3(20)(7)(20 + 7)$	1	1	3	4	0	$3(7)(20)(20 + 7)$	1	1	3	4	0
$(27)^3$	1	9	6	8	3	$(27)^3$	1	9	6	8	3

Rule-5: Cube of a square number– Cube of given square number is the square of the cube of square-root of the number. Algebraic expression is $\left\{\left(\sqrt{a}\right)^3\right\}^2 = (a)^3$.

Example: Square number is 9. $\left\{\left(\sqrt{9}\right)^3\right\}^2 = \{3^3\}^2 = \{27\}^2 = 729 = 9^3$.

Generalisation of square and cube of a numbers:

Let us see how Bhāskarācārya find the square of a number.

a) A two-digit number $(ab)$is a Binomial, where $(ab) = 10a + b$

$(ab)^2 = (10a + b)^2 = 100a^2 + 20ab + b^2$

This can be written as:

$(ab)^2$		
10^2	10	1
a^2	$2ab$	b^2

The Binomial terms a^2 $2ab$ b^2 are arranged from right to left, in group of 2, in ascending powers of 10.

b) A Trinomial (abc) is reduced to a Binomial (xc) we get $(xc)^2 = x^2$ $2xc$ c^2 in order of power of 10 where x being in the binomial $10a + b$. Then, the trinomial $(abc) = 100a + 10b + c$.

Arranging in a group of 2 we get:

$(abc)^2$				
10^4	10^3	10^2	10^2	1
a^2	$2ab$	b^2	$2(ab)c$	c^2

where(ab) as before stands for the binomial $10a + b$ in 10's place.

c) Similarly writing $(abcd)$ as binomial (yd) where $y = (abc)$, we have by successive reduction and arranging the binomial terms in group of 2: $(abcd)^2 = (abc)^2$ $2(abc)d$ d^2 where $abcd = 1000a + 100b + 10c + d$

It can be expressed as:

$(abcd)^2$						
10^6	10^5	10^4	10^3	10^2	10	1
a^2	$2ab$	b^2	$2(ab)c$	c^2	$2(abc)d$	d^2

d) Extending this to cubes we have $(ab)^3,\ (abc)^3,\ (abcd)^3$ arranged in a group of 3, we get

$(ab)^3$			
10^3	10^2	10	1
a^3	$3a^2b$	$3ab^2$	b^3

$(abc)^3$						
10^6	10^5	10^4	10^3	10^2	10	1
a^3	$3a^2b$	$3ab^2$	b^3	$3(ab)^2c$	$3(ab)c^2$	c^3

$(abcd)^3$									
10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2	10	1
a^3	$3a^2b$	$3ab^2$	b^3	$3(ab)^2c$	$3(ab)c^2$	c^3	$3(abc)^2d$	$3(abc)d^2$	d^3

Quadratic Equation

तृतीयोऽध्यायः।पञ्चमःपरिच्छेद।

अथगुणकर्म।तत्रदृष्टमूलजातौकरणसूत्रंवृत्तद्वयम्।

गुणघ्नमूलोनयुतस्यराशेर्दृष्टस्ययुक्तस्यगुणार्द्धकृत्या।

मूलंगुणार्द्धेनयुतंविहीनंवर्गीकृतंप्रष्टुरभीष्टराशिः॥६२

Transcription

Tṛtīyo'dhyāyaḥ | Pañcamaḥ pariccheda |

Atha guṇakarma | Tatra dṛṣṭamūlajātau karaṇsutraṁ vṛttadvayam |
Guṇaghnamūlonayutasya rāśerdṛṣṭasya yuktasya guṇārddhakṛtyā |
Mūlaṁ guṇārddhena yutaṁ bihīnaṁ vargīkṛtaṁ praṣṭurabhīṣṭarāśiḥ || 62
English version
To sum or difference of a quantity and a multiple of square-root of the quantity is given:
$x^2 \pm bx$ where x^2 is the quantity to be determined and b is multiple or co-efficient.
Square of half the co-efficient is added to the quantity: $x^2 \pm bx + \frac{b^2}{4}$.
Square-root of the sum is extracted: $\sqrt{x^2 \pm bx + \frac{b^2}{4}} = \sqrt{\left(x \pm \frac{b}{2}\right)^2} = x \pm \frac{b}{2}$
Half the co-efficient is added or subtracted then squared:

$$\left(x \pm \frac{b}{2} \mp \frac{b}{2}\right)^2 = x^2$$

यदालवैश्चोनयुतश्चराशिरेकेनभागोनयुतेनभक्त्वा।
दृश्यंतथामूलगुणश्चताभ्यांसाभ्यस्ततःप्रोक्तवदेवराशिः॥६३
Transcription
Yadā lavaiśconayutaśca rāśirekena bhāgonayutena bhaktvā |
Dṛśyaṁ tathā mūlaguṇaśca tābhyāṁ sābhyastataḥ proktavadeva rāśiḥ || 63
English version
If the quantity have fraction (of itself) added or subtracted:
$x^2 \pm \frac{1}{n}x^2 \pm bx = \left(1 \pm \frac{1}{n}\right)x^2 \pm bx.$
Divide the number given and the multiplicator of the root increasing or decreasing the fraction by unity: $1 \pm \frac{1}{n}$ i.e. multiplying by $\frac{1}{1 \pm \frac{1}{n}}$. Then it becomes: $x^2 \pm b\left(\frac{1}{1 \pm \frac{1}{n}}\right)x = x^2 \pm cx$ where $c = b\left(\frac{1}{1 \pm \frac{1}{n}}\right)$ is new coefficient. Now proceed with the equation $x^2 \pm cx$ as before to get the result.
योराशिःमूलेनकेनचिहनुणितेनऊनोदृष्टस्तस्यगुणार्द्धकृत्यायुक्तस्यदृष्टस्ययत्पदंतद्गुणार्द्धेनयुक्तंकार्य्यंयदिगुणघ्नमूलयुतोदृष्टस्तर्हिहीनंकार्य्यंतस्यवर्गोराशिस्यात्।
Transcription
Yo rāśiḥ mūlena kenacihnuṇitena ūno dṛṣṭastasya guṇārddhakṛtyā yuktasya dṛṣṭasya yatpadaṁ tadguṇārddhena yuktaṁ kāryyaṁ yadi guṇaghnamūlayuto dṛṣṭastarhi hinaṁ kāryya tasya vargorāśi syāt|
English version
This third paragraph is not verse but prose and it explained the above two verses. Symbolically we can express as:
Suppose the quantity is *x* and the equation is $x \pm a\sqrt{x} = b$ --- --- (1)
The, completing the square, we get:

$$x \pm a\sqrt{x} + \left(\frac{a}{2}\right)^2 = b + \left(\frac{a}{2}\right)^2$$

$$\therefore \sqrt{x} = \sqrt{b + \left(\frac{a}{2}\right)^2} \mp \frac{a}{2}$$

$$\therefore x = \left\{\sqrt{b + \left(\frac{a}{2}\right)^2} \mp \frac{a}{2}\right\}^2$$

Hence, the reason for the first part of the rule is clear. It is the ordinary rule for solving an equation reducible to a quadratic by completing the square.
The second part of the rule is meant for equations of the form:
$x \pm \frac{c}{d}x \pm a\sqrt{x} = b$ --- (2)

When we get $x \pm \frac{a\sqrt{x}}{1 \pm \frac{c}{d}} = \frac{b}{1 \pm \frac{c}{d}}$ which is of the form (1) and may be solved as above.

मूलेनदृष्टेतावदुदाहरणम्।
बालेमरालकुलमूलदलानिसप्ततीरेविलासभरमन्थरगाण्यपश्यम्।
कुर्वच्चकेलिकलहंकलहंसयुग्मंशेषंजलेवदमरालकुलप्रमाणम्॥६४
न्यासः।मूलगुणः $\frac{७}{२}$;दृश्यम।२।; दृष्टस्यास्य।२।;गुणार्द्धकृत्या $\frac{४९}{१६}$; युक्तास्य $\frac{८१}{१६}$; मूलम् $\frac{९}{४}$; गुणार्द्धेन $\frac{७}{४}$; युतम्।४।वर्गकृतंजातंहंसकुलमानम्।१६।

Transcription

Mūlena dṛṣṭe tāvadudāharaṇam |
Bāle marālakulamūladalāni sapta tire vilāsabharamantharagāṇyapaśyam |
Kurvacca kelikalahaṁ kalahaṁsayugmaṁ śeṣaṁ jale vada marālakulapramāṇam || 64
Nyāsaḥ | Mūlaguṇaḥ $\frac{7}{2}$; Dṛśyama | 2 |; Dṛṣṭasyāsya | 2 |; Guṇārddhakṛtyā $\frac{49}{16}$; Yuktāsya $\frac{81}{16}$; Mūlam $\frac{9}{4}$; Guārddhena $\frac{7}{4}$; Yutam | 4 |; Vargakṛtaṁ jātaṁ haṁsakulamānam | 16 |

Example of quadratic equation with involvement of square-root:

Dear girl! There was a flock of swans on a lakeside. Seven and half times of the square-root of number of swans in the flock are proceeding towards the shore due to tired of the diversion. One pair of geese is sporting in the water. Tell me the number of swans in the flock.

Solution

Statement: Co-efficient of square-root of number of flock is $\frac{7}{2}$. Residual number in flock is 2. Square of half of co-efficient = $\left(\frac{7}{4}\right)^2 = \frac{49}{16}$. Add with residual number = $2 + \frac{49}{16} = \frac{81}{16}$. Its square-root = $\frac{9}{4}$. Half the co-efficient is added with = $\frac{9}{4} + \frac{7}{4} = \frac{16}{4} = 4$. Square of it is 16 = Number swans in flock.

Algebraic symbolically: Let the number of swans in the flock = x

Then, $2 + \frac{7}{2}\sqrt{x} = x$

$\therefore x - \frac{7}{2}\sqrt{x} + \left(\frac{7}{4}\right)^2 = 2 + \left(\frac{7}{4}\right)^2 = \left(\frac{9}{4}\right)^2$; Or, $\left(\sqrt{x} - \frac{7}{4}\right)^2 = \left(\frac{9}{4}\right)^2$;

Or, $\sqrt{x} - \frac{7}{4} = \frac{9}{4}$; Therefore, $x = \left(\frac{7}{4} + \frac{9}{4}\right)^2 = 16$.

{If we consider, number of swan in the flock = x^2; then equation will be $2x^2 - 7x - 4 = 0$}

अथमूलयुतेदृष्टेउदाहरणम्।
खपदैर्नवभिर्युक्तंस्याच्चत्वारिंशताधिकम्।
शतद्वादशकंविद्वन्कःसराशिर्निगद्यताम्॥६५
न्यासः।;मूलगुणः।९।;दृश्यम।१२४०।; गुणार्द्धेन$\frac{९}{२}$; कृत्या$\frac{८९}{४}$; युतंजातम्$\frac{५०४१}{४}$; अस्यमूलं$\frac{७१}{२}$; गुणार्द्धेन$\frac{९}{२}$; अत्रविहीनम्।३१। ; वर्गीकृतंजातोराशिः।९६१।

Transcription

Atha mūlayute dṛṣṭe udāharaṇam |
Khapadairnavabhiryuktaṁ syāccatvāriṁśtādhikam |
Śatadvādaśakaṁ vidvan kaḥsa rāśirnigadyatām || 65
Nyāsaḥ |; Mūlaguṇaḥ | 9 |; Dṛśyama | 1240 |; Guṇārddhena $\frac{9}{2}$; Kṛtyā $\frac{81}{4}$; Yutaṁ jātam $\frac{5041}{4}$; Asya Mūlaṁ $\frac{71}{2}$; Guṇārddhena $\frac{9}{2}$; Atra vihīnam | 31 |; Vargikṛtaṁ jāto rāśiḥ |961 |

Now example of addition of square-root:

Solution

Oh! Learned person, root is added and the sum is given. Tell me what will be the number when it is added with nine-times its square root amounts twelve hundred and forty.

Statement: Co-efficient is 9. Total value is 1240. Half of the co-efficient is $\frac{9}{2}$. Square of it = $\left(\frac{9}{2}\right)^2 = \frac{81}{4}$. Adding with the amount = 1240 + $\frac{81}{4} = \frac{5041}{4}$. Square-root of it = $\frac{71}{2}$. Subtracting $\frac{9}{2}$ from it = $\frac{71}{2} - \frac{9}{2} = 31$. Squaring it = 961 and it is the answer.
Algebraic symbolically: Let the number is *x*.
We have to solve $x + 9\sqrt{x} = 1240$
$\therefore \left(\sqrt{x}\right)^2 + 2 \times \frac{9}{2}\sqrt{x} + \frac{81}{4} = 1240 + \frac{81}{4} = \frac{5041}{4}$.
Or, $\left(\sqrt{x} + \frac{9}{2}\right)^2 = \left(\frac{71}{2}\right)^2$
Or, $\sqrt{x} = \frac{71}{2} - \frac{9}{2} = 31$
Therefore, $x = 961$
{If we consider, number = x^2; then equation will be $x^2 + 9x - 1240 = 0$}
भागमूलोनेदृष्टउदाहरणम्।
जातंहंसकुलस्यमूलदशकंमेघागमेमानसंप्रोड्डोयस्थलपद्मिनीवनमगादष्टांशकोऽनम्मस्तटात्।
वालेवालमृणालशालिनिजलेकेलिक्रियालालसंदृष्टंहंसयुगत्रय़ञ्चसकलांयुथस्यवद॥६६
न्यासः।;मूलगुणः।१०।;भागः $\frac{१}{८}$;दृश्यम।६।;यदालवैश्चनयुतइत्युक्तत्वादचैकेनभागेनेन $\frac{७}{८}$; दृश्यमूलगुणौभक्ताजातंदृश्य $\frac{४८}{७}$; मूलगुणः $\frac{८०}{७}$; गुणार्द्धम् $\frac{४०}{७}$; अस्यकृत्या $\frac{१६००}{४९}$; युक्तंदृश्यम् $\frac{१९३६}{४९}$; अस्यमूलं $\frac{४४}{७}$; गुणार्द्धेन $\frac{४०}{७}$; युतंवर्गीकृतंजातोहंसराशिः।१४४।
Transcription
Bhāgamūlone dṛṣṭa udāharaṇam |
Jātaṁ haṁsakulasya mūladaśakaṁ meghāgame mānasaṁ
Proḍḍoya sthalapadminīvanamagādaṣṭāṁśako'mmastaṭāt |
Bāle bālamṛṇālaśālini jale kelikriyālālasaṁ dṛṣṭaṁ
Haṁsayugatrayañca sakalāṁ yuthasya bada || 66
Nyāsaḥ |; Mūlaguṇaḥ | 10 |; Bhāgaḥ $\frac{1}{8}$; Dṛśama | 6 |; Yadā lavaiścanayuta ityuktatyādacaikena bhāgenena $\frac{7}{8}$; Dṛśyamūlaguṇau bhaktā jātaṁ dṛśya $\frac{48}{7}$; Mūlaguṇa $\frac{80}{7}$; Guṇārddhaḥ $\frac{40}{7}$; Asya kṛtyā $\frac{1600}{49}$; Yuktaṁ dṛśyam $\frac{1936}{49}$; Asya mūlaṁ $\frac{44}{7}$; Guṇārddhena $\frac{40}{7}$; Yutaṁ vargīkṛtaṁ jāto haṁsarāśiḥ |144 |
Example of subtraction of root as well as fraction:
Solution
Of the flock of geese at *Mānasasarovara*[21] ten times of the square-root of the number of geese departed on the approach of cloud and eighth part of the number in flock went to the forest *Sthalapadmini*[22] forest. Three couples were seen engaged in sport on the water abounding with delicate fibres of lotus. Dear girl, tell me the number of geese in the flock.
Statement: Co-efficient of square-root of the number in flock = 10. Dividing number in flock by 8 or multiplying by $\frac{1}{8}$. Remaining number is 6. Less the fraction from unity = $\frac{7}{8}$. Residual number divided by this fraction $6 \div \frac{7}{8} = \frac{48}{7}$. Dividing co-efficient by this fraction = $10 \div \frac{7}{8} = \frac{80}{7}$. Half of it = $\frac{40}{7}$; Squaring it = $\frac{1600}{49}$. Find by addition = $\frac{1600}{49} + \frac{48}{7} = \frac{1936}{49}$; Square-root of it = $\frac{44}{7}$; Half of the co-efficient divided by $\frac{7}{8} = \frac{40}{7}$. Adding and squaring we get the number = $\left(\frac{44}{7} + \frac{40}{7}\right)^2 = 144$.
Algebraically: Let *x* is the whole number of geese in the flock.
Then, $10\sqrt{x} + \frac{1}{8}x + 6 = x$
Or, $\frac{7}{8}x - 10\sqrt{x} = 6$
Or, $x - \frac{80}{7}\sqrt{x} = \frac{48}{7}$

Or, $\left(\sqrt{x}-\frac{40}{7}\right)^2=\frac{48}{7}+\frac{1600}{49}=\frac{1936}{49}=\left(\frac{44}{7}\right)^2$

Or, $x=\left(\frac{40}{7}+\frac{44}{7}\right)^2=(12)^2=144$

{If we consider, number of geese in Mānosarovara = x^2
then equation will be $7x^2 - 80x - 48 = 0$}

उदाहरणम्।
पार्थःकर्णबधायमर्गणगणंक्रुद्धोरणेसन्दधे
तस्यार्द्धेननिवार्य्यतच्छरगणंमूलैश्चतुर्भिर्हयान्।
शल्यंषड्िलरथेषुभिस्त्रिभिरपिच्छत्रंध्वजंकार्मुकं
चिच्छेदास्यशिरःशरेणकतितेयान्अर्जुनःसन्दधे॥६७
न्यासः।; मूलगुणः।४।;भागः$\frac{१}{२}$;दृश्यम।१०।; यदालवैश्चोनयुतइत्यादिनाजातंबानमानम्॥१००॥

Transcription
Udāharaṇam |
Pārthaḥ Karṇabadhāya margaṇagaṇaṁ kruddho raṇe sandadhe
Tasyārddhena nivāryya taccharagaṇaṁ mūlaiścaturbhirhayān |
Śalyaṁ ṣaḍliratheṣubhistribhirapi cchatraṁ dhvajaṁ kārmukaṁ
Cicchedāsya śiraḥ śareṇa kati te yān Arjunaḥ sandadhe || 67
Nyāsaḥ |; Mūlaguṇaḥ | 4 |; Bhāgaḥ $\frac{1}{2}$; Dṛśyama |10 |; Yadā lavaiśconayuta ityādinā jātaṁ bānamānam || 100 ||

Example:
Solution:
Pārtha[23], son of Prithā[24], irritated to be furious in war to kill Karṇa[25] and took a quiver of arrows. With half of his arrows he destroyed all of the Karṇa's arrows. He killed Karṇa's horses with four-times of square-root of number of arrows in quiver. With six arrows he slew Salya[26] (spear). He used one arrow each to destroy the top of the chariot, the flag and the bow of Karṇa. Finally, he cut off Karṇa's head with another arrow. How many of the arrows which Arjuna let fly?

Statement: Co-efficient is 4. Fraction is $\frac{1}{2}$. Number of other used arrows = 10.

Algebraically: Let x denote the number of used by Arjuna.

Then, equation will be: $\frac{1}{2}x+4\sqrt{x}+6+3+1=x$

Or, $x-\frac{x}{2}-4\sqrt{x}=10$

Or, $x-8\sqrt{x}=20$

Or, $\left(\sqrt{x}-4\right)^2=20+16=36=6^2$

Or, $\sqrt{x}=4+6=10$

Or, $x=100$

{If we consider, number of arrows in quiver of Pārtha = x^2;
then equation will be $x^2 - 8x - 20 = 0$}

अपिच।
अलिकुलदलमूलंमालतींयातमष्टौनिखिलनवमभागाश्चालिनीमृङ्गमेकम्।
निशिपरिमललुब्धंपद्ममध्येनिरुद्धंप्रतिरणतिरणन्तंब्रूहिकान्तेऽलिसङ्ख्याम्॥६८
अत्रकिलराशिनवांशाकंराश्यर्द्धमूलश्चराशेर्ऋणंरुपहयंदृश्यम्।एतदृणंदृश्यञ्चार्द्धितंराश्यर्द्धस्यभवतीति।
तथान्यासः।;मूलगुणकः$\frac{१}{२}$; भागः$\frac{८}{९}$;दृश्यम।१।; प्राग्वल्लब्धंराशिदलम्।३६।एतदद्विगुणितमलिकुलमानम्।७२।

Transcription
Apica |

Alikuladadalamūlaṁ mālatiṁ yātamaṣṭau
nikhilanavamabhāgāścālinī mṛṅgamekam |
Niśi parimalaluvdhaṁ padmamadhye niruddhaṁ
pratiraṇati raṇantaṁ brūhi kānte'lisaṅkhyām || 68
Atra kila rāśinavāṁśākaṁ rāśyarddhamūlaśca rāśerṛṇaṁ rupahayaṁ dṛśyaṁ |
Etadṛṇaṁdṛśyañcārddhitaṁrāśyarddhasyabhavatīti |
Tathānyāsaḥ | ; Mūlaguṅakaḥ $\frac{1}{2}$; Bhāgaḥ $\frac{8}{9}$; Dṛśyama | 1 |; Prāgvallavdhaṁ rāśidalam | 36 | Etadadviguṇitamalikulamānam |72 |

Solution

Square-root of half the number of a swarm of black bees[27] is gone to a shrub of Mālatī[28]. Again eight-ninths of the whole swarm went to Mālatī tree. A female is buzzing[29] to one remaining male that is humming within a lotus in which he is captivated, having been allured to it by its fragrance at night[30]. He started wailing and his beloved responded. Say lovely woman, the number of bees.

Here eight-ninths of the quantity and the root of its half are negative i.e. to be subtracted from the quantity and remaining number of bees is two. The negative quantity and the given number halved bring out half the quantity sought.

Thus, statement: Number of bees multiplied by $\frac{1}{2}$. Take $\frac{8}{9}th$ part of whole number of bees. Find half the given number = 1. A fraction of half the quantity is the same as half the fraction of the quantity. Proceeding as above directed half the quantity = 36. Considering double of it number of bees in the swarm = 72.

Algebraically, let us take number of bees within the swarm = x

Then, $\sqrt{\frac{1}{2}x} + \frac{8}{9}x + 2 = x$

Now, put $x = 2y$ and we get

$2y - \frac{16}{9}y - \sqrt{y} = 2$

Or, $\frac{2}{9}y - \sqrt{y} = 2$

Or, $y - \frac{9}{2}\sqrt{y} = 9$

Or, $\left(\sqrt{y} - \frac{9}{4}\right)^2 = 9 + \frac{81}{16} = \frac{225}{16} = \left(\frac{15}{4}\right)^2$

Or, $\sqrt{y} = \frac{9}{4} + \frac{15}{4} = 6$

Or, $y = 6^2 = 36$

Therefore, number of black bees in the swarm = $x = 2y = 2 \times 36 = 72$

{If we consider, number of black bees = $2x^2$; then equation will be $2x^2 - 9x - 18 = 0$}

भागमूलयुतेदृष्टेउदाहरणम्।
योराशिदशभिःस्वमूलैःराशित्रिभागेणसमन्वितश्च।
जातंशतद्वादशकंतमाशुजानीहिपाट्यांपटुतास्तितेचेत॥६९
न्यासः।;मूलगुणः।१८।;भागः$\frac{१}{३}$;दृश्यम।१२००।; अनैकेनभागयुतेन$\frac{४}{३}$; मूलगुणंदृश्यश्चभक्त्वाप्राग्वज्वातोराशिः।५७६।

Transcription

Bhagamūlayute dṛṣṭe udāharaṇam |
Yo rāśidaśabhi svamūlaiḥ rāśitribhāgeṇa samanvitaśca |
Jātaṁ śatadvādaśakaṁ tamāśu jānīhi pāṭyāṁ paṭutāsti te ceta || 69
Nyāsaḥ |; Mūlaguṇaḥ | 18 |; Bhāgaḥ $\frac{1}{3}$; Dṛśyama | 1200 |; Anaikena bhāgayutena $\frac{4}{3}$; Mūlaguṇaṁ dṛśyaśca bhaktvāprāgvajvātorāśiḥ | 576 |

Example of problem with addition of fraction and root:

Solution

If you have a skill in arithmetic, find out quickly the number; if eighteen times of its square-root and one-third of it be added to it yield twelve hundred.

Statement: Square-root of number is multiplied by 18. Dividing the number by 3. Sum becomes 1200. Adding get $\frac{4}{3}$ times we get result = 576.

Algebraically, let the number be *x* then, $x + \frac{1}{3}x + 18\sqrt{x} = 1200$

Or, $\frac{4}{3}x + 18\sqrt{x} = 1200$

Or, $x + \frac{27}{2}\sqrt{x} = 900$

Or, $\left(\sqrt{x} + \frac{27}{4}\right)^2 = 900 + \left(\frac{27}{4}\right)^2 = 900 + \frac{729}{16} = \frac{15129}{16} = \left(\frac{123}{4}\right)^2$

Or, $\sqrt{x} = \frac{123}{4} - \frac{27}{4} = 24$ Therefore, $x = 576$

{If we consider the number = x^2; then equation will be $2x^2 - 27x - 1800 = 0$}

Transcription / Transliteration

अ	A, a	आ	Ā, ā	इ	I, i	ई	Ī, ī
उ	U, u	ऊ	Ū, ū	ऋ	Ṛ, ṛ	ए	E, e
ऐ	ai	ओ	o	औ	ou	क	K, k
ख	Kh, kh	ग	G, g	घ	Gh. gh	ङ	ṅ
च	C, c	छ	Ch, ch	ज	J, j	झ	Jh, jh
ञ	ñ	ट	Ṭ ṭ	ठ	Ṭh, ṭh	ड	Ḍ, ḍ
ढ	Ḍh, ḍh	ण	ṇ	त	T, t	थ	Th, th
द	D, d	ध	Dh, dh	न	N, n	प	P, p
फ	Ph, ph	ब	B, b	भ	Bh, bh	म	M, m
य	y	र	R, r	ल	L, l	व	V, v
श	Ś, ś	ष	Ṣ, ṣ	स	S, s	ह	H, h
ड़	ḋ	ढ़	ḋh	य़	y̤	ऽ	’
ं	ṁ	ः	ḥ	ऌ	Ḷ, ḷ		

RESULTS AND DISCUSSION

Result

People frequently use square & cube the numbers for calculating area & volume those methods have been simplified by *Bhāskarācārya* through play time algebra.

In our higher-secondary school level quadratic equation is used for problem solving on throwing of a ball, shooting an arrow, firing a missile etc. Ultimately these slow down and come back on earth. The whole process can be visualized through *Quadratic Equations.* Calculations are on the basis of primary operations. Those methods were preached by Indian Mathematical stalwarts amongst whom *Bhāskarācārya* played a prominent role.

Conclusion

All the above described processes, even today, using in school level mathematics. This indicates how updated was *Bhāskara-II*. More over *Bhāskarācārya* has expressed the ideas in mathematical model i.e. in symbolic mood though the algebraic symbols were not in used then. The above dissertation leads us to *how Bhāskarācārya solved the arithmetical problems on the basis of algebraic mood. Bhāskarācārya's* work *Siddhānta Śiromaṇīḥ* particularly *Līlāvatī,* a fascinating example of *Mathematics in verses,* is relevance even today.

At the end we can quote the Śloka from Līlāvatī of Bhāskarācārya:

येषांसुजातिगुणवर्गविभूषिताङ्गीशुद्धाखिलव्यवहृतिःखलुकण्टसक्ता।

लीलावतीहसरसोक्तिमुदाहरन्तीतेषांसदैवसुखसम्पदुपैतिवृद्धिम्॥२७७

Transcription

Yeṣāṁ sujātiguṇavargavibhūṣitāṅgi śuddhākhilavyavahṛtiḥ khalu kaṇṭasaktā |
Līlāvatīha sarasoktimudāharantī teṣāṁ sadaiva sukhasampadupaiti vṛddhim || 277

Those who have memorized and studied the text *Līlāvatī*, whose ornaments are the interesting illustrations of the division, multiplication, squares and all types of day-to-day flawless calculations etc., will indeed be ever happy and prosperous in this world. In India *Līlāvatī* used to teach for nearly 500 years as mathematics course in school level; so, it has rich mathematical educational value.

Notes

[1]It is an excellent example of concert between a difficult subject like mathematics and verses. *Bhāskarācārya*'s *Līlāvatī* and *Bījagaṇitam* used to teach in India for about more than 500 years. No other textbook has enjoyed such long lifespan. Bhāskarācārya took good parts of *Śridharācārya*'s *Triśatikā* and *Mahāvīrācārya*'s *Gaṇitasārasaṁgraha.* He wrote a commentary on *Sidhānta Śiromaniḥ* named as *Vāsanābhāṣya sahitaḥ* of *Mitākṣara* and also wrote different books with titles: *Karaṇa-Grantha, Jātaka-Tīkā-Grantha*, *Karaṇkutuhala, Sarvatobhadrayantra, Vaisiṣthatulya* and *Vivāhapaṭala.*

Literally *Līlāvatī*means "Beautiful" or "Playful". It has been speculated that *Līlāvatī* was *Bhāskarācārya*'s own daughter but no such evidence in his writings. Thus *Līlāvatī* is probably, a mere name, describing the mood pervading the mathematical literature with glimpses of beautiful flights of imagination and poetic ingenuity. *Līlāvatī* had obtained important scientific book, particularly mathematical book, among the scholars of Middle-East. Persian translation of *Līlāvatī* was patronised by Mughal Emperor *Ākbar* and translated by renown scholar of his court *Abū-al-Fayḍ-Fayḍī* in 1587.

Līlāvatī, seemingly the name of a female to whom instruction is addressed. But the term is interpreted in some of the commentaries, consistently its etymology, *charming.*

[2]*Pāṭī-gaṇitam* or *Vyakta- gaṇitam* where Pāṭī comes from *Paripāṭī* means method and *Gaṇita* means calculation.

[3]*Bījagaṇitam* is the generic name for Algebra where Bīja means seed or root. So, *Bījagaṇitam* is calculation on seeds which potentially contains calculus on manifestation of numbers. Another name of *Bījagaṇitam* is *Avyakta-Gaṇita* i.e. calculus of non-manifested numbers i.e. unknowns.

[4] 1114CE – 1185CE, the place of his residence was identified as *Vijjaladvīd and* he was also known as *Bhāskara-II*.

[5]गणकचक्रचूडामणिi.e. "AGa the name was designated by "Jewel among the mathematicians ṇeśa Daivajña and (CE 1507)Literally means *supreme of counting as well as formulating mathematical & astronomical concepts*.

[6] This verse found in *Pandit Jivānanda Vidyāsāgara*'s edition of *Līlāvatī*. This verse was probably added with *Līlāvatī* by some pupil of *Bhāskarācārya*.

[7]Literally means *Crest-Jewel of Siddhāntas*.

[8]Aindra (or Indra), Chandragomin, Kāsakṛtṣna, Āpiśāli, Śākaṭāyana, Pāṇini, Amara Sinha and Ācārya Pūjapada (or Pūjapāda or Jainendra) are the Vyākaraṇīs or Grammarians.

[9] Agnivesa-Saṁhitā, Bheda-Saṁhitā, Jātūkarma-Saṁhitā, Parāsara-Saṁhitā, Sīrapāni-Saṁhitā and Hārīta-Saṁhitā. This six ancient works were the basis of works of Caraka, Śuśruta and Bāgvata. The works of Caraka &Suśruta are called Saṁhitās whereas that of Bāgvata is known as Aṣtāṁga Hṛdaya.

[10] Sāṅkhya, Yoga, Nyāya, Vaiśeṣika, Mīmāṁsā and Vedānta.

[11]Pouliśa Siddhānta, Romaka Siddhānta, Vaśiṣtha Siddhānta and Paitāmahā Siddhānta.

[12] Ṛk, Yajur, Sām and Atharva.

[13] Three Prasthānas of Hindu Religious Philosophy. These are collectively thee points of departure as Upaniṣads, Vāgbat Gītā & Brahma Sūtra. 1) Śruti Prasthāna is hearing of Upaniṣads i.e. dialogues between Guru & Śiṣya (Teacher & student); 2) Smriti Prasthāna is to be remembered from Vāgbat Gītā; 3) Brahma Sūtraswhich Nyāya Prasthāna which set forth to explain logically all the doctrines taught from the Upaniṣads and Vāgbat Gītā.

[14]Pūrva-mimāṁsā &Uttar-mimāṁsā whereas Pūrva-mimāṁsā of Jaimini is called Mimāṁsā (deals with Dharma, Kartavya of in Vedas & Upaniṣads) Uttar-mimāṁsā of Vyāsa is known as Vedānta (deals with spiritual & philosophical themes of Upaniṣads). These taught Eternal Brahmas and the aim & scope of both.
[15] 505 CE – 587 CE from Avanti (modern Malwa) wrote Bṛhat Saṁhitā, Pañca Siddhāntikā contained Sūrya Siddhānta (Astronomical concepts on Sun), Pouliśa Siddhānta (Astronomical concepts Pouliśa i.e. Greek from the city of Saintra i.e. Alexandria), Romaka Siddhānta (Astronomical concepts of Rome), Vaśiṣtha Siddhānta (Astronomical concepts of Star Great Bear), Brahma Siddhānta / Paitāmahā Siddhānta (Astronomical concepts of universe).
[16] 598 CE – c.665 CE from Billamata (modern Bhinmal of Rajasthan) wrote Brāmhasphuṭa Siddhānta Astronomical concepts of Bramha i.e. universe, Khaṇḍakhādyaka (Astronomical concepts of practical application).
[17] Planetary motion or Mathematical Astronomy
[18] Astronomy on Sphere.
[19] Born at Ostenfelde: 31 October, 1815 – Died at Berlin: 19 February, 1897. He is known as *father of modern analysis.* He was an expert in elliptic & Abelian functions and developed irrational numbers theory.
[20] In a letter to Sofia Kovalevskaya on 27 August, 1883 as shared by Gösta Mittag-Leffler at the 2nd International Congress in Paris. Published in *Compte rendu du deuxième Congrès international des mathematiciens tenu à Paris du 6 au 12 août 1900,* Gauthier-Villars (Paris), 1902, page.149.

- Sofia Vesilyevna Kovalevskaya (Born: Moscow; 15 January, 1850 – Died: Sweden; 10 February, 1891), first Russian female mathematician, contributed on Mathematical Analysis & Mechanics. She was an editor of international scientific journal.
- Magnus Gustaf (Gösta) Mittag-Leffler (Born: Stockholm; 16 March, 1846 – Died: Djursholm; 7th July, 1927), Swedish Mathematician, worked on Theory of Functions.

[21] Lake.
[22] Land-Lily or Land-Lotus.
[23] Arjuna, surnamed Bartha; his matronymic from Prithā.
[24] Kunti
[25] Hero of Kauravas warriors.
[26] One of the Kauravas, and charioteer of Karṇa.
[27]अलि:, Aliḥ; some-one consider it to be भ्रमरor Humble-bee.
[28] Jasmin, jasminum grandiflorum.
[29]गुंजनं; Humming.
[30] Lotus being open at night and closed in the day, the bees might be caught in it.

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