

REDUCTION IN A TRIADIC FORMAL CONTEXT

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ABSTRACT

Triadic concept analysis is an extension of formal concept analysis. Generally, for a triadic formal context, some attributes (objects or condition) may not be essential in triadic concept. Thus, this paper studies the reduction from three aspects: attribute reduction that keeps object-condition unchanged, object reduction that keeps attribute-condition unchanged, condition reduction that keeps object-attribute unchanged, and the corresponding algorithms are given respectively.

Keywords: *triadic concept, triadic formal context, consistent set, reduction*

INTRODUCTION

Formal concept analysis (FCA) was proposed by Ganter and Wille (1982). FCA has been evolved into an efficient methodology for data analysis. Based on the idea of three domain, Lehmann and Wille, (1995) extended FCA to the triadic formal context, proposed triadic concept analysis (TCA), two key components in TCA include triadic formal context and triadic concept. Triadic formal context is a quadruple which contains object set, attribute set, condition set and a ternary relation between them. A triadic concept is determined by a triple, which contains an object subset (extent), an attribute subset (intent) and a condition subset (modus). Due to the growing number of three-dimensional data on the network, such as the core data structure of social resource sharing systems Folksonomy, includes three sets of users, tags and resources, and a ternary relation between them, it is very suitable for analysis using TCA. As a new information processing theory, due to its specificity, has been applied to various fields in the social resource sharing systems, such as Flickr system, bookmarking system, bibsonomy system. Therefore, the related research has become a hot topic in artificial intelligence field, for example, Belohlavek and Osicka, (2012) studied triadic concept from two dimensions, and proposed triadic concept of data with grade attribute, and then, Belohlavek *et al.*, (2013) studied optimal factorization using triadic concept, Wang *et al.*, (2017) combine rough set theory into TCA. Moreover, Wei *et al.*, (2014, 2016) summarizes the research progress of triadic concept lattices, from the review, we know TCA still have a lot of work to do.

The same as FCA, the amount of data and the high computational complexity is one of the problems faced by TCA, thus the study of reduction in a formal context is a meaningful research topic. For a triadic formal context, we introduce the reduction from three aspects: attribute, object, condition, while keeping the original concept unchanged.

The remainder of this paper is organized as follows. Firstly, we briefly review some basic notions of TCA. Secondly, we will discuss the reduction from three aspects, and then the corresponding algorithms are given respectively. Finally, concludes the paper and outlines the future work.

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Preliminaries

In this section, we recall some basic notions of TCA. Please refer to Lehmann and Wille, (1995), Wille,(1995) for detailed information.

Definition 1 (1) (Lehmann and Wille, 1995) A triadic context is defined as a quadruple $\mathbb{K} = (K_1, K_2, K_3, Y)$, where $K_1 = \{g_1, \dots, g_p\}$ is a non-empty finite set of objects, $K_2 = \{m_1, \dots, m_q\}$ is a non-empty finite set of attributes, and $K_3 = \{b_1, \dots, b_j\}$ is a non-empty finite set of conditions, Y is a ternary relation between K_1 , K_2 and K_3 , i.e. $Y \subseteq K_1 \times K_2 \times K_3$. And $(g, m, b) \in Y$ is read: the object g has the attribute m under the condition b .

(2) (Lehmann and Wille, 1995) Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, for $\{i, j, k\} = \{1, 2, 3\}$ with $j < k$, $X \subseteq K_i, Z \subseteq K_j \times K_k$, the (i) -derivation operators are defined by

$$X \mapsto X^{(i)} := \{(a_j, a_k) \in K_j \times K_k \mid \text{For } \forall a_i \in K_i, (a_i, a_j, a_k) \in Y\},$$

$$Z \mapsto Z^{(i)} := \{a_i \in K_i \mid \text{For } \forall (a_j, a_k) \in Z, (a_i, a_j, a_k) \in Y\}.$$

it is evident that $\mathbb{K}^{(i)} = (K_i, K_j \times K_k, Y^{(i)})$, i.e., for $\forall g \in K_i, m \in K_j, b \in K_k$, we have $gY^{(1)}(m, b) \Leftrightarrow gY^{(2)}(m, b) \Leftrightarrow gY^{(3)}(m, b) \Leftrightarrow (g, m, b) \in Y$.

Lehmann and Wille,(1995) defined the further derivation operators as follows:

(3) (Lehmann and Wille, 1995) Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, for $\{i, j, k\} = \{1, 2, 3\}$ and for $X_i \subseteq K_i, X_j \subseteq K_j$, and $X_k \subseteq K_k$, the (i, j, X_k) -derivation operators are defined by

$$X_i \mapsto X_i^{(i, j, X_k)} := \{a_j \in K_j \mid \text{For } \forall (a_i, a_k) \in X_i \times X_k, (a_i, a_j, a_k) \in Y\},$$

$$X_j \mapsto X_j^{(i, j, X_k)} := \{a_i \in K_i \mid \text{For } \forall (a_j, a_k) \in X_j \times X_k, (a_i, a_j, a_k) \in Y\}.$$

In fact, the operator is equivalent to the operator in the formal context $\mathbb{K}_{X_k}^{ij} = (K_i, K_j, Y_{X_k}^{ij})$, where

$$(a_i, a_j) \in Y_{X_k}^{ij} \Leftrightarrow \forall a_k \in X_k, (a_i, a_j, a_k) \in Y.$$

Example 1. Let $(i, j, k) = (1, 2, 3)$, $(a_1, a_2) \in Y_{X_3}^{12}$ is read: the object a_1 has the attribute a_2 under all condition a_3 with $a_3 \in X_3$.

Definition 2 (Lehmann and Wille, 1995) Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, for $\{i, j, k\} = \{1, 2, 3\}$ with $j < k$, and $X_i \subseteq K_i, X_j \subseteq K_j$, and $X_k \subseteq K_k$, if satisfying $X_i = (X_j \times X_k)^{(i)}, (X_i, X_j, X_k)$ is called a triadic concept, the components X_i , X_j , and X_k are called the extent, the intent, and the modus of (X_i, X_j, X_k) , respectively.

the set of all triadic concepts of (K_1, K_2, K_3, Y) is denoted by $\mathfrak{S}(\mathbb{K})$.

So, we can obtain $\mathfrak{S}(\mathbb{K})$, where $\mathbb{K}$ is a triadic formal context by the Definition 2.

Example 2. Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, which is $K_1 = \{A, B, C\}$, $K_2 = \{1, 2, 3, 4, 5, 6\}$, $K_3 = \{a, b, c, d\}$.

A: Superordinate stereotypes	B: Unspecified	C: Subtypes
1: Physical appearance	2: Political beliefs	3: Attitudes
4: Behavior		
5: Traits	6: Situations	
a: Recall	b: Impression Formation	c: Behavior Prediction
		d: Evaluation

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Table 1. A triadic context (K_1, K_2, K_3, Y)

a	b						c						d					
	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6
A	×	×					×	×				×	×	×		×		
B	×										×		×					×
C	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×			×

According to the Definition 2, we can obtain $\mathfrak{I}(\mathbb{K}) = \{(\emptyset, 123456, abcd), (A, 124, d), (B, 15, cd), (C, 126, abcd), (C, 123456, ab), (AC, 12, abcd), (AC, 126, c), (BC, 6, b), (ABC, 1, acd), (ABC, 123456, \emptyset), (ABC, \emptyset, abcd)\}$.

Reduction methods and reduction algorithms

In this section, we will describe the idea of reduction from three aspects. At first, we introduce attribute reduction that keeps object-condition unchanged in triadic concept.

Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, its attribute sub-context is referred to $\mathbb{K}' = (K_1, K_2', K_3, Y')$, where $K_2' \subseteq K_2$, and $Y' = Y \cap (K_1 \times K_2' \times K_3)$.

Definition 3 (1) Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, for any $(X_1, X_2, X_3) \in \mathfrak{I}(\mathbb{K})$, there exist $X_2' \subseteq K_2'$, such that $(X_1, X_2', X_3) \in \mathfrak{I}(\mathbb{K}')$, where $\mathbb{K}'$ is the attribute sub-context of $\mathbb{K}$. K_2' is called a attribute consistent set of K_2 .

(2) Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, for $x \in K_2$, if $K_2 - \{x\}$ is a consistent set of K_2 , x is called a dispensable attribute; otherwise, x is indispensable attribute of K_2 .

(3) Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, K_2' is a consistent set of K_2 , and for any $x \in K_2', K_2' - \{x\}$ is not a consistent set of K_2 , furthermore, K_2' is called an attribute reduction of $\mathbb{K}$.

Theorem 1: Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, K_2' is an attribute reduction of $\mathbb{K}$ if and only if $x \in K_2'$ for all x is indispensable attribute.

Proof:(Sufficiency) Since K_2' is an attribute reduction of $\mathbb{K}$. According to the Definition 3 (3), we have K_2' is a consistent set of K_2 , and for any $x \in K_2', K_2' - \{x\}$ is not a consistent set of K_2 , that is to say, for all $x \in K_2'$ is indispensable attribute by the Definition 3 (2).

(Necessity) First, we write set $R = \{x \in K_2 | x \text{ is indispensable attribute}\}$. Suppose R is not a consistent set of K_2 . Due to $R \subseteq K_2$, in other words, $R \cup \{x_1\} \cdots \{x_j\} = K_2$, according to the Definition 3 (3), we must have $\{x_j\} \in K_2$, but $\{x_j\} \notin R$, x_j is a dispensable attribute, which contradicts the condition. Therefore, R is a consistent set of K_2 . Furthermore, for any $x \in R$, due to x is a indispensable attribute, thus, $R - \{x\}$ is not a consistent set of R , according to the Definition 3 (3), we have R is an attribute reduction of $\mathbb{K}$, i.e., $K_2' = R$.

Theorem 1 shows that an attribute reduction of $\mathbb{K}$ can be obtained by indispensable attribute, the corresponding algorithm is summarized as follows.

Algorithm 1. Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context.

Input: a triadic formal context $\mathbb{K} = (K_1, K_2, K_3, Y)$.

Output: an attribute reduction R' .

Step 1 initialize $R' = \emptyset$;

Step 2 compute $\mathfrak{I}(\mathbb{K})$;

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Step 3 for every $\{x\} \in K_2$, $K_2' = K_2 - \{x\}$, compute $\mathfrak{T}(\mathbb{K}')$, where $\mathbb{K}' = (K_1, K_2', K_3, Y')$;
 Step 4 compare $\mathfrak{T}(\mathbb{K})$ with $\mathfrak{T}(\mathbb{K}')$, if $\{x\}$ is indispensable attribute, $R' = R' \cup \{x\}$;
 otherwise, $R' = R'$;
 Step 5 output R' .

Example 3. We use the context in Example 2 to examine algorithm 1.

1. According to Definition 2, we can obtain $\mathfrak{T}(\mathbb{K}) = \{(\emptyset, 123456, abcd), (A, 124, d), (B, 15, cd), (C, 126, abcd), (C, 123456, ab), (AC, 12, abcd), (AC, 126, c), (BC, 6, b), (ABC, 1, acd), (ABC, 123456, \emptyset), (ABC, \emptyset, abcd)\}$.

2. Due to $K_1 = \{A, B, C\}$, $K_2 = \{1, 2, 3, 4, 5, 6\}$, $K_3 = \{a, b, c, d\}$, according to the Definition 3 (2), it can be checked that $K_2' = K_2 - \{1\}$, and by the Definition 2, we can obtain $\mathfrak{T}(\mathbb{K}') = \{(\emptyset, 23456, abcd), (A, 24, d), (B, 5, cd), (C, 23456, ab), (C, 26, abcd), (AC, 2, abcd), (AC, 26, c), (BC, 6, b), (ABC, 23456, \emptyset), (ABC, \emptyset, abcd)\}$.

by Definition 3 (1) and (2), it not keeping object-modus unchanged, that is to say, $\{1\}$ is a indispensable attribute.

Similarly, since $K_2' = K_2 - \{2\}$, according to the Definition 2, we can obtain $\mathfrak{T}(\mathbb{K}') = \{(\emptyset, 13456, abcd), (A, 14, d), (B, 15, cd), (C, 13456, ab), (C, 16, abcd), (AC, 1, abcd), (AC, 16, c), (BC, 6, b), (ABC, 1, acd), (ABC, \emptyset, abcd), (ABC, 13456, \emptyset)\}$.

by Definition 3 (1) and (2), it can keep object-modus unchanged, that is to say, $K_2' = K_2 - \{2\}$ is a consistent set of K_2 , i.e., $\{2\}$ is a dispensable attribute.

Similarly, In turn to check $K_2' = K_2 - \{3\}$, $K_2 - \{4\}$, $K_2 - \{5\}$, $K_2 - \{6\}$, we can know, $\{3\}$ is a dispensable attribute, $\{4\}$, $\{5\}$, $\{6\}$ both are indispensable attribute.

In a word, in Example 3, attribute $\{1\}$, $\{4\}$, $\{5\}$, $\{6\}$ is indispensable attribute, attribute $\{2\}$, $\{3\}$ is dispensable attribute. Furthermore, according to Definition 3 (3), we can know $\{1, 4, 5, 6\}$ is an attribute reduction of $\mathbb{K}$.

Secondly, we will introduce the idea of object reduction that keeps attribute-condition unchanged, due to the proof of related theorem is the same to those theorem 1, for brevity, we omit the proof.

Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, its object sub-context is referred to $\mathbb{K}' = (K_1', K_2, K_3, Y')$, where $K_1' \subseteq K_1$, and $Y' = Y \cap (K_1' \times K_2 \times K_3)$.

Definition 4 (1) Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, for any $(X_1, X_2, X_3) \in \mathfrak{T}(\mathbb{K})$, there exist $X_1' \subseteq K_1'$, such that $(X_1', X_2, X_3) \in \mathfrak{T}(\mathbb{K}')$, where $\mathbb{K}'$ is the object sub-context of $\mathbb{K}$. K_1' is called an object consistent set of K_1 .

(2) Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, for $A \in K_1$, if $K_1 - \{A\}$ is an object consistent set of K_1 , A is called a dispensable object; otherwise, A is indispensable object of K_1 .

(3) Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, K_1' is a object consistent set of K_1 , and for any $A \in K_1$, $K_1' - \{A\}$ is not a object consistent set of K_1 , furthermore, K_1' is called an object reduction of $\mathbb{K}$.

Theorem 2: Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, K_1' is an object reduction of $\mathbb{K}$ if and only if $A \in K_1'$ for all A is indispensable object.

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Analogously, the algorithm for object reduction is summarized as follows.

Algorithm 2. Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context.

Input: a triadic formal context $\mathbb{K} = (K_1, K_2, K_3, Y)$.

Output: an object reduction R'' .

Step 1 initialize $R'' = \emptyset$;

Step 2 compute $\mathfrak{I}(\mathbb{K})$;

Step 3 for every $\{A\} \in K_1$, $K_1' = K_1 - \{A\}$, compute $\mathfrak{I}(\mathbb{K}'')$, where $\mathbb{K}'' = (K_1'', K_2, K_3, Y'')$;

Step 4 compare $\mathfrak{I}(\mathbb{K})$ with $\mathfrak{I}(\mathbb{K}'')$, if $\{A\}$ is indispensable object, $R'' = R'' \cup \{A\}$;

otherwise, $R'' = R''$;

Step 5 output R'' .

Example 4. We use the context in Example 2 to examine algorithm 2.

1. According to Definition 2, we can obtain $\mathfrak{I}(\mathbb{K}) = \{(\emptyset, 123456, abcd), (A, 124, d), (B, 15, cd), (C, 126, abcd), (C, 123456, ab), (AC, 12, abcd), (AC, 126, c), (BC, 6, b), (ABC, 1, acd), (ABC, 123456, \emptyset), (ABC, \emptyset, abcd)\}$.

2. According to the Definition 4 (2), it can be checked that $K_1' = K_1 - \{A\}$, and by the Definition 2, we can obtain $\mathfrak{I}(\mathbb{K}'') = \{(\emptyset, 123456, abcd), (B, 15, cd), (C, 126, abcd), (C, 123456, ab), (BC, 6, b), (BC, 1, acd), (BC, 123456, \emptyset), (BC, \emptyset, abcd)\}$. Obviously, we have K_1' is not a consistent set of K_1 by the Definition 4 (1), furthermore, according to the Definition 4 (2), we have $\{A\}$ is a indispensable object.

Similarly, in turn to check that $K_1' = K_1 - \{B\}$, $K_1' = K_1 - \{C\}$, we can know, $\{B\}, \{C\}$ both are indispensable object.

In a word, in Example 4, object $\{A\}, \{B\}, \{C\}$ both are indispensable object.

Furthermore, according to the Definition 4 (3), we have $\{A, B, C\}$ is an object reduction of $\mathbb{K}$.

Thirdly, the idea of condition reduction that keeps object-attribute unchanged will be introduced, due to the proof of related theorem is the same to those theorem 1, for brevity, we omit the proof.

Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, its condition sub-context is referred to $\mathbb{K}' = (K_1, K_2, K_3', Y''')$, where $K_3' \subseteq K_3$, and $Y''' = Y \cap (K_3 \times K_2 \times K_3')$.

Definition 5 (1) Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, for any $(X_1, X_2, X_3) \in \mathfrak{I}(\mathbb{K})$, there exist $X_3' \subseteq K_3'$, such that $(X_1, X_2, X_3') \in \mathfrak{I}(\mathbb{K}''')$, where $\mathbb{K}'''$ is the condition sub-context of $\mathbb{K}$. K_3' is called a condition consistent set of K_3 .

(2) Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, for $a \in K_3$, if $K_3 - \{a\}$ is a condition consistent set of K_3 , a is called a dispensable condition; otherwise, a is indispensable condition of K_3 .

(3) Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, K_3' is a condition consistent set of K_3 , and for any $a \in K_3'$, $K_3' - \{a\}$ is not a condition consistent set of K_3 , furthermore, K_3' is called a condition reduction of $\mathbb{K}$.

Theorem 3: Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context, K_3' is a condition reduction of $\mathbb{K}$ if and only if $a \in K_3'$ for all a is indispensable condition.

Analogously, the algorithm for condition reduction is summarized as follows.

Algorithm 2. Let $\mathbb{K} = (K_1, K_2, K_3, Y)$ be a triadic formal context.

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Input: a triadic formal context $\mathbb{K} = (K_1, K_2, K_3, Y)$.

Output: a condition reduction R''' .

Step 1 initialize $R''' = \emptyset$;

Step 2 compute $\mathfrak{Z}(\mathbb{K})$;

Step 3 for every $\{a\} \in K_3$, $K_3' = K_3 - \{a\}$, compute $\mathfrak{Z}(\mathbb{K}''')$, where $\mathbb{K}''' = (K_1, K_2, K_3', Y''')$;

Step 4 compare $\mathfrak{Z}(\mathbb{K})$ with $\mathfrak{Z}(\mathbb{K}''')$, if $\{a\}$ is indispensable condition, $R''' = R''' \cup \{a\}$;

otherwise, $R''' = R'''$;

Step 5 output R''' .

Example 5. We use the context in Example 2 to examine algorithm 3.

1. According to Definition 2, we can obtain $\mathfrak{Z}(\mathbb{K}) = \{(\emptyset, 123456, abcd), (A, 124, d), (B, 15, cd), (C, 126, abcd), (C, 123456, ab), (AC, 12, abcd), (AC, 126, c), (BC, 6, b), (ABC, 1, acd), (ABC, 123456, \emptyset), (ABC, \emptyset, abcd)\}$.

2. Due to $K_1 = \{A, B, C\}$, $K_2 = \{1, 2, 3, 4, 5, 6\}$, $K_3 = \{a, b, c, d\}$, according to the Definition 5 (2), it can be checked that $K_3' = K_3 - \{a\}$, and by the Definition 2, we can obtain $\mathfrak{Z}(\mathbb{K}''') = \{(\emptyset, 123456, bcd), (A, 124, d), (B, 15, cd), (C, 126, bcd), (C, 123456, b), (AC, 12, bcd), (AC, 126, c), (BC, 6, b), (ABC, 1, cd), (ABC, 123456, \emptyset), (ABC, \emptyset, bcd)\}$. By the Definition 5 (1) and (2), obviously, it keeping object-attribute unchanged, that is to say, $\{a\}$ is a dispensable condition.

Similarly, in turn to check that $K_3' = K_3 - \{b\}$, $K_3 - \{c\}$, $K_3 - \{d\}$, we have, $\{b\}$, $\{c\}$, $\{d\}$ both are indispensable condition.

In a word, according to Theorem 3, we can know $\{b, c, d\}$ is a condition reduction of $\mathbb{K}$.

Conclusion

In this paper, we describe the idea of reduction from three aspects, in case of keeping triadic concept unchanged, making the triadic formal context more concise. How to construct the triadic concept lattice more effectively is an important work of TCA, we will focus on the parallel construction algorithm of TCA in the future.

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