Three-way concept in a generalized one-sided formal context

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Abstract
In this paper, we firstly introduce one-sided three-way operators (OS-operators) and one-sided three-way concepts (OS-concepts), which are based on positive attributes, negative attributes, positive operators and negative operators. Next we discuss properties of OS-operators and OS-concepts in detail. Then, we propose one algorithm to obtain all OS-concepts in a generalized one-sided formal context.

Keywords: One-Sided Formal Context, Operator, Three-Way Concept Lattice

Introduction
Formal concept analysis (FCA) was proposed by Ganter and Wille, (1999), and has become an important tool for data analysis and knowledge mining. As the core structure of FCA, formal concept lattice has been widely concerned. Formal concept lattice reflects all the hierarchical knowledge of formal concepts, which establish an object-attribute model in formal context. Initially, FCA was based on a formal context, where the binary relation between a set of objects and a set of attributes was characterized by 0 or 1. In real life, truth value is not always binary. According to different formal contexts, many researchers search for ways to obtain concepts. For example, Kent, (1994) presented rough concept analysis; Burusco and Fuentes-González, (2000) introduced the fuzzy theory to the FCA; Yahia and Jaoua, (2001) proposed one-sided fuzzy concept; In order to deal with many-value formal context, Butka and Pócs, (2013) put forward a generalized one-sided formal context, and based on Galois connections, proposed a generalized one-sided concept lattices; Halaš and Pócs, (2015) proposed attribute preferences for generalized one-sided concept lattices; Shao and Li, (2015) proposed a new type of concept and attribute reduction in a generalized one-sided formal context.

Three-way decision (3WD) theory was proposed by Yao (2012), which adds a third option (non-commitment) on the basis of the two-way decision model. Combining the 3WD and FCA, Qi et al., (2014) proposed three-way analysis (3WCA), which extended the classical FCA. Three-way concepts provide a novel object-attribute model. Different from a formal concept, the extension (or intension) of a three-way concept is included two parts, which are “jointly possessed” and “jointly not possessed”. Based on a three-way concept, one can divide objects (or attributes) into three regions to make three-way decisions.

Since no work has been done on three-way concept analysis in a generalized one-sided formal context, it is interesting to obtain three-way concepts in such a context. Firstly, we divide the attribute set into two parts: positive attributes and negative attributes. Combing the positive operator and negative operator, we will introduce one-sided three-way operators, and discuss their properties. We will also propose a one-sided three-way concept, and present the corresponding algorithm.

The remainder of this paper is organized as follows. Firstly, we briefly review some basic notions of concept lattice in a generalized one-sided formal context. Secondly, we propose OS-operators (one-sided three-way operators) and OS-concepts (one-sided three-way concepts) in a generalized one-sided formal context, discuss their properties, and propose one algorithm to obtain all OS-concepts. Finally, concludes the paper.

Preliminaries
In this section, we briefly give some basic notions related to generalized one-sided formal context and one-sided concept. For more details please refer to Butka and Pócs, (2013) and Shao and Li, (2015).

Throughout this paper, we write $P(S)$ as the power set of $S$, which is a non-finite set, $DP(S) = P(S) \times P(S)$. 

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For $(A, B), (C, D) \in DP(S)$, we define $(A, B) \cap (C, D) = (A \cap B, C \cap D), (A, B) \cup (C, D) = (A \cup B, C \cup D)$, and $(A, B) \supseteq (C, D)$ if $A \subseteq C, D \subseteq B$.

As for many-valued contexts, Butka and Pócs (2013) proposed a generalized one-sided formal context.

**Definition 1** (Butka and Pócs, 2013) A 4-tuple $\mathbb{K} = (U, A, L, R)$ is said to be a generalized one-sided formal context if the following conditions are fulfilled:

1. $U$ is a non-empty set of objects and $A$ is a non-empty set of attributes.
2. $L: A \rightarrow CL$ is a mapping from the set of attributes $A$ to the class of all complete $CL$. Hence, for any attribute $a \in A$, $L(a)$ denotes a structure of truth values for attribute $a$.
3. $R$ is a generalized incidence relation, i.e., $R(x, a) \in L(a)$ for all $x \in U$ and $a \in A$. Thus, $R(x, a)$ represents a degree from the structure $L(a)$ in which the element has the attribute $a$.

**Example 1.** A generalized one-sided formal context $(U, A, L, R)$, and generalized incidence relation $R$ is defined in Table 1, where $U = \{x_1, x_2, x_3, x_4\}$, $A = \{a, b, c\}$.

<table>
<thead>
<tr>
<th>$R$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
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<tbody>
<tr>
<td>$x_1$</td>
<td>2</td>
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<td>$x_2$</td>
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<tr>
<td>$x_3$</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Be similar to the formal concept in formal context, Butka and Pócs, (2013) gave a pair of operators and obtained one-sided concepts.

**Definition 2** (Butka et al., 2013): Let $(U, A, L, R)$ be a generalized one-sided formal context. A pair of mappings $\nabla: P(U) \rightarrow \prod_{a \in A} L(a)$ and $\Delta: \prod_{a \in A} L(a) \rightarrow P(U)$ are defined for $X \in P(U)$ and $g \in \prod_{a \in A} L(a)$ as follows:

- $X^\nabla(a) = \land_{x \in X} R(x, a), a \in A$.
- $g^\Delta = \{ x \in U | \forall a \in A, g(a) \leq R(x, a) \}$.

Since properties of the two operators in Definition 2 have not been discussed in Butka and Pócs (2013), we give a detailed analysis about these properties.

**Proposition 1:** Let $(U, A, L, R)$ be a generalized one-sided formal context. $X, X_1, X_2, X_i \in P(U)$ and $g, g_1, g_2, g_i \in \prod_{a \in A} L(a)$. Then

1. $X_1 \subseteq X_2 \Rightarrow X_1^\nabla \subseteq X_2^\nabla$, and $g_1 \leq g_2 \Rightarrow g_1^\Delta \subseteq g_2^\Delta$.
2. $X \subseteq X^\Delta \nabla$, and $g \leq g^\nabla \Delta$.
3. $X^\nabla = X^\nabla \Delta$, and $g^\Delta = g^\Delta \nabla \Delta$.
4. $(X_i \nabla)^\nabla = \land_{i \in I} X_i^\nabla$, and $(\land_{i \in I} g_i)^\Delta = \land_{i \in I} g_i^\Delta$, where $I$ is an index set.

**Proof:** (1) According to the Definition 2, we have $X^\nabla(a) = \land_{x \in X} R(x, a), a \in A$. Since $X_1 \subseteq X_2$, we know $\land_{x \in X_1} R(x, a) = (\land_{x \in X_1} R(x, a)) \wedge (\land_{x \in (X_2 - X_1)} R(x, a)) \leq \land_{x \in X_1} R(x, a), i.e., X_1^\nabla \subseteq X_2^\nabla$.

Analogously, we have $g_1^\Delta = \{ x \in U | \forall a \in A, g_1(a) \leq R(x, a) \}$ by Definition 2. Due to $g_1 \leq g_2$, so for any $x \in U$, we have $g_1(a) \leq g_2(a) \leq R(x, a)$. Therefore, $g_1^\Delta \subseteq g_2^\Delta = \{ x \in U | \forall a \in A, g_1(a) \leq R(x, a) \}$.

(2) According to Definition 2, we have $X^\nabla \Delta = \{ x \in U | \forall a \in A, X^\nabla(a) \leq R(x, a) \} = \{ x \in U | \forall a \in A, \land_{x \in X} R(x, a) \leq R(x, a) \}$. Due to for each $x \in X$ and $a \in A$, it is easy to obtain that $\land_{x \in X} R(x, a) \leq
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R(x,a), so we have X \subseteq X^{\Delta A}.

Similarly, according to Definition 2, for each a \in A, we have 
\[ g^{\Delta A}(a) = \land_{x \in g^{-1}} R(x,a) \geq \land_{g(a) \leq R(x,a)} R(x,a) \geq g(a). \]
Therefore, we have g \leq g^{\Delta A}.

(3) On the one hand, from (1) and (2), we have X^{\Delta U} \leq X^V. On the other hand, X^V \in \prod_{a \in A} \mathcal{L}(a), from (2), we have X^V \leq X^{\Delta U}. To sum up, we have X^V = X^{\Delta U}.

Analogously, on the one hand, from (1) and (2), we have g^{\Delta U} \leq g^V. On the other hand, g^V \in 2^U, from (2), we have g^V \leq g^{\Delta U}. To sum up, we have g^V = g^{\Delta U}.

(4) According to Definition 2, it is easy to obtain.

Definition 3 (Butka and Pócs, 2013) Let (U,A,L,R) be a generalized one-sided formal context. A pair (X,g) is called a generalized one-sided concept if it satisfies X^V = g and g^A = X, where X \in P(U), g \in \prod_{a \in A} \mathcal{L}(a), and the set X is referred to as the extension and g as the intension of the concept (X,g).

Now let us see Example 1. Take X = \{x_1,x_2,x_4\} and g = (2,0,1). It is not difficult to check that X^V = (2,0,1) = g, g^A = \{x_1,x_2,x_4\} = X.

That is, (X,g) is called a generalized one-sided concept.

Remark 1: To avoid confusion, we called the above operators \lor and \land as one-sided negative operators. We call a concept, which is induced by one-sided negative operators, as negative one-sided concept (for short, NO-concept). The set of all pairs (X,g) is denote by OL(U,A,L,R).

Lemma 1: (Butka and Pócs, 2013) Let (U,A,L,R) be a generalized one-sided formal context. Then \(\forall \mathcal{L}(U,A,L,R)\) with the partial order defined above forms a complete lattice, where 
\[ \land_{i \in I} (X_i,g_i) = (\bigcap_{i \in I} X_i, (\bigvee_{i \in I} g_i)^{\Delta A}), \quad \lor_{i \in I} (X_i,g_i) = ((\bigcup_{i \in I} X_i)^{\Delta U}, \land_{i \in I} g_i). \]

Analogously, Shao and Li, (2015) discussed a generalized one-sided formal context, gave a pair of operator (correspondingly, called one-sided positive operators) and obtain a new type of concept in a generalized one-sided formal context.

Definition 4: (Shao and Li, 2015) Let (U,B,L,R) be a generalized one-sided formal context. A pair of mappings \(\uparrow: P(U) \to \prod_{b \in B} \mathcal{L}(b)\) and \(\downarrow: \prod_{b \in B} \mathcal{L}(b) \to P(U)\) are defined for X \in P(U) and f \in \prod_{b \in B} \mathcal{L}(b) as follows:

\[ X^\uparrow(b) = \lor_{x \in X} R(x,b), b \in B. \]

A pair (X,f) is called a PO-concept if X^\uparrow = f and f^\downarrow = X, where X is called the extension and f is called intension of (X,f). In particular, both (X^\uparrow, X^\downarrow) and (f^\downarrow, f^\uparrow) are PO-concepts.

Now let us see Example 1. Take X = \{x_1,x_2,x_3\} and f = (3,1,1). It is not difficult to check that X^\uparrow = (3,1,1) = f, f^\downarrow = \{x_1,x_2,x_3\} = X. Thus, (X,f) is called a PO-concept.

The family of all the PO-concepts can forms a complete lattice denoted by POL(U,B,L,R), which is called the PO-concept lattice of (U,B,L,R).

\[ \lor_{(X_1,f_1)}(X_2,f_2) \in POL(U,B,L,R), \text{ then the partial order is defined as follows:} \]

\[ (X_1,f_1) \preceq (X_2,f_2) \iff X_1 \subseteq X_2 \iff f_1 \preceq f_2. \]

Where, (X_1,f_1) is referred to as sub-concept of (X_2,f_2), and (X_2,f_2) is referred to as super-concept of (X_1,f_1).

Lemma 2 (Shao and Li, 2015) Let \(\mathbb{K} = (U,B,L,R)\) be a generalized one-sided formal context. Then, POL(U,B,L,R) equipped with the partial order \preceq forms a complete lattice in which the meet and the join...
Generalized One-Sided Three-Way Concept

In real life, there exists a special type of many-value formal context (see Table 2), the relationship between attributes and objects is not consistent: A part of the attribute is positively related to the object, that is to say, for an object set, the larger the value of the attribute the better; Another part of the attribute is negatively related to the object, that is, the smaller the value of the attribute the better. According to the meaning of attributes, we always divide all attributes into two classes denoted by A and B. A denotes the set of negative attributes, which are more meaningful when their values are more smaller, and B denotes the set of positive attribute, which are more meaningful when their values are larger. Therefore, in this section, combining positive attributes, negative attributes, negative operators and positive operators, we try to obtain a type of generalized one-sided three-way concept.

Remark 2: Let $\mathcal{K} = (U, A \cup B, \mathcal{L}, R)$ be a generalized one-sided formal context, where $A$ is the set of negative attributes, $B$ is the set of positive attributes, $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$, $\mathcal{L}_1: A \to CL$, and $\mathcal{L}_2: B \to CL$.

Example 2: From a hospital database, we choose 4 patients i.e., $U = \{x_1, x_2, x_3, x_4\}$ and 7 attributes denoted as $\{a, b, c, d, e, m, n\}$, which represents blood sugar, blood lipids, blood pressure, amino acid, macro element, microelement, vitamins, respectively. “−” means normal levels of the human body, “+” means a measure that exceeds the normal level of the human body.

Firstly, let us analyses these attributes. Note that the lower of the values of blood sugar, blood lipids, blood pressure, it is more meaningful for the human healthy body. Meanwhile, within a certain range, the higher of the values of amino acid, macro element, microelement, vitamins, it is also meaningful for the human healthy body. Therefore, we choose $A = \{a, b, c\}$ and $B = \{d, e, m, n\}$.

Table 2: A Medical Measurement Background

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>m</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>++</td>
<td>-</td>
<td>+</td>
<td>++</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$x_2$</td>
<td>+++</td>
<td>+</td>
<td>+</td>
<td>+++</td>
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<td>+++</td>
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<tr>
<td>$x_3$</td>
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<td>+</td>
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<td>++</td>
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<tr>
<td>$x_4$</td>
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</tbody>
</table>

Next, we obtain that $\mathcal{L}_1(a) = 4, \mathcal{L}_1(b) = 3, \mathcal{L}_1(c) = 2, \mathcal{L}_2(d) = 4, \mathcal{L}_2(e) = 4, \mathcal{L}_2(m) = 3,$ and $\mathcal{L}_2(n) = 4$, where 4,3,2,4,4,3, and 4 denotes four, three, two, and four, four, three, and four elements chain, respectively.

Then a generalized incidence relation $R$ is captured in the following table:

Table 3: A Generalized One-Sided Formal Context

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>D</th>
<th>e</th>
<th>m</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
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<tr>
<td>$x_2$</td>
<td>3</td>
<td>1</td>
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<td>3</td>
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<td>2</td>
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<td>$x_3$</td>
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<td>0</td>
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<td>2</td>
</tr>
<tr>
<td>$x_4$</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
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**Definition 5:** Let $\mathbb{K} = (U, A \cup B, L, R)$ be a generalized one-sided formal context, $X \subseteq P(U), g \in \Pi_{A \in A} \lambda_A (a), \text{and } f \in \Pi_{B \in B} \lambda_B (b)$. A pair of mappings $\sqsupseteq: P(U) \rightarrow \Pi_{A \in A} \lambda_A (a) \times \Pi_{B \in B} \lambda_B (b)$ and $\sqsubseteq: \Pi_{A \in A} \lambda_A (a) \times \Pi_{B \in B} \lambda_B (b) \rightarrow P(U)$ are defined by $X^\sqsupseteq = (X^\sqsupseteq, X^\sqsubseteq)$. 

A pair of mappings $(g, f) \in \mathbb{K}$ is defined by $(g, f)^\sqsupseteq = \{x \in U | \forall a \in A, g(a) \leq R(x, a) \text{ and } \forall b \in B, f(b) \geq R(x, b)\} = g^A \cap f^1$. 

We abbreviate them as one-sided three-way operators (in short, OS-operators).

**Proposition 3:** Let $\mathbb{K} = (U, A \cup B, L, R)$ be a generalized one-sided formal context, $X, X_1, X_2, X_i \in P(U)$ and $g, g_1, g_2, g_i, f, f_1, f_2, f_i \in \Pi_{A \in A} \lambda_A (a), \Pi_{B \in B} \lambda_B (b)$, then

(OS1) $X \subseteq X_i \Rightarrow X_1^\sqsubseteq \subseteq X_2^\sqsubseteq$, and $(g, f_1) \subseteq (g_1, f_2) \Rightarrow (g_1, f_1)^\sqsubseteq \subseteq (g_2, f_2)^\sqsubseteq$.

(OS2) $X \subseteq X_i \Rightarrow (g, f) \supseteq (g, f_i)^\sqsupseteq$.

(OS3) $X^\sqsubseteq = X^\sqsubseteq^\sqsupseteq$, and $(g, f)^\sqsubseteq = (g, f)^\sqsubseteq^\sqsupseteq$.

(OS4) $X \subseteq (g, f)^\sqsubseteq \Rightarrow X^\sqsubseteq \subseteq (g, f)$.

(OS5) $(X_1 \cup X_2)^\sqsubseteq \supseteq X_1^\sqsubseteq \cup X_2^\sqsubseteq$, and $((g_1, f_1) \cup (g_2, f_2))^\sqsubseteq \supseteq (g_1, f_1)^\sqsubseteq \cup (g_2, f_2)^\sqsubseteq$.

(OS6) $(X_1 \cap X_2)^\sqsubseteq \subseteq X_1^\sqsubseteq \cap X_2^\sqsubseteq$, and $((g_1, f_1) \cap (g_2, f_2))^\sqsubseteq \subseteq (g_1, f_1)^\sqsubseteq \cap (g_2, f_2)^\sqsubseteq$.

**Proof:** (OS1) Since $X_1 \subseteq X_2$, from Proposition 1 and Proposition 2, we have $X_1^\sqsubseteq \subseteq X_2^\sqsubseteq$ and $X_1^\sqsubseteq \subseteq X_2^\sqsubseteq$. According to Definition 5, we have $X_1^\sqsubseteq = (X_1^\sqsubseteq, X_1^\sqsubseteq) \subseteq (X_2^\sqsubseteq, X_2^\sqsubseteq) = X_2^\sqsubseteq$.

Similarly, since $(g_1, f_1) \subseteq (g_2, f_2)$, we know that $g_2 \leq g_1$ and $f_1 \leq f_2$. Then, from Proposition 1 and Proposition 2, we have $g_1^A \subseteq g_2^A$ and $f_1^1 \subseteq f_2^1$. Furthermore, by Definition 5, we have $(g_1, f_1)^\sqsubseteq = (g_1^A \cap f_1^1) \subseteq (g_2^A \cap f_2^1) = (g_2, f_2)^\sqsubseteq$.

(OS2) According to Definition 5, we have $X^\sqsubseteq \subseteq X_1^\sqsubseteq \cap X_2^\sqsubseteq$. From Proposition 1 and Proposition 2, we know $X \subseteq X_1^\sqsubseteq$ and $X \subseteq X_2^\sqsubseteq$, thus, we get $X \subseteq (X_1^\sqsubseteq \cap X_2^\sqsubseteq) = X^\sqsubseteq$.

Analogously, by Definition 5, we have $(g, f)^\sqsubseteq = (g^A \cap f^1)^\sqsubseteq = (g^A \cap f^1)^\sqsubseteq \cap (g^A \cap f^1)^\sqsubseteq = g^A \cap f_1^1 \subseteq f_1^1$, from Proposition 1 and 2, we know $g \leq g^A$ and $f \leq f^1$, thus we get $(g, f)^\sqsubseteq = (g^A \cap f^1)^\sqsubseteq \cap (g^A \cap f^1)^\sqsubseteq \subseteq (g, f)^\sqsubseteq$.

(OS3) On the one hand, from Proposition (OS2), we have $(X^\sqsubseteq) \subseteq (X^\sqsubseteq)^\sqsubseteq$. On the other hand, from Proposition (OS2), we know $X \subseteq X^\sqsubseteq$, and according to Proposition (OS1), we have $X^\sqsubseteq \subseteq X^\sqsubseteq^\sqsupseteq$. Thus we obtain $X^\sqsubseteq = X^\sqsubseteq^\sqsupseteq$.

Similarly, on the one hand, from Proposition (OS2), we have $((g, f)^\sqsubseteq) \subseteq ((g, f)^\sqsubseteq)^\sqsupseteq$. On the other hand, from Proposition (OS2), we know $(g, f) \subseteq (g, f)^\sqsubseteq$, and according to Proposition (OS1), we have $((g, f)^\sqsubseteq) \subseteq ((g, f)^\sqsubseteq)^\sqsupseteq$. Thus we obtain $((g, f)^\sqsubseteq) = (g, f)^\sqsubseteq$.

(OS4) **Necessity:** Since $X \subseteq (g, f)^\sqsubseteq$, from Proposition (OS1), we have $X^\sqsubseteq \subseteq (g, f)^\sqsubseteq$. Furthermore, from Proposition (OS2), we have $(g, f)^\sqsubseteq \subseteq (g, f)$, thus, we get $X^\sqsubseteq \subseteq ((g, f)^\sqsubseteq) \subseteq (g, f)$.

**Sufficiency:** Since $X^\sqsubseteq = (X_1^\sqsubseteq, X_2^\sqsubseteq) \subseteq (g, f)^\sqsubseteq$, by Definition 5 and (OS1), we know $X^\sqsubseteq = \subseteq (g, f)^\sqsubseteq$. From Proposition (OS2), we have $X \subseteq X^\sqsubseteq$, Thus, we get that $X \subseteq X^\sqsubseteq \subseteq (g, f)^\sqsubseteq$.

(OS5) According to Definition 5, we have $(X_1 \cup X_2)^\sqsubseteq = ((X_1 \cup X_2)^\sqsubseteq, (X_1 \cup X_2)^\sqsubseteq)$, $X_1^\sqsubseteq = (X_1^\sqsubseteq, X_1^\sqsubseteq)$ and $X_2^\sqsubseteq = (X_2^\sqsubseteq, X_2^\sqsubseteq)$. Furthermore, from Proposition 1 and Proposition 2, we know $X_1^\sqsubseteq \subseteq X_2^\sqsubseteq = (X_1^\sqsubseteq, X_1^\sqsubseteq)$ and $X_2^\sqsubseteq = (X_2^\sqsubseteq, X_2^\sqsubseteq)$. Furthermore, from Proposition 1 and Proposition 2, we know $(X_1 \cup X_2)^\sqsubseteq = (X_1^\sqsubseteq, X_1^\sqsubseteq)$ and $(X_2 \cup X_2)^\sqsubseteq = (X_2^\sqsubseteq, X_2^\sqsubseteq)$. To sum up, we can know $(X_1 \cup X_2)^\sqsubseteq = (X_1^\sqsubseteq, X_1^\sqsubseteq)$ is equivalent to $(X_1^\sqsubseteq \cup X_2^\sqsubseteq, X_1^\sqsubseteq \cup X_2^\sqsubseteq) = X_1^\sqsubseteq \cup X_2^\sqsubseteq$. That is to say, $(X_1 \cup X_2)^\sqsubseteq \subseteq X_1^\sqsubseteq \cup X_2^\sqsubseteq$.

Analogously, by Definition 5, we have $((g_1, f_1) \cup (g_2, f_2))^\sqsubseteq = ((g_1 \cup g_2)^A \cap (f_1 \cup f_2)^1)^\sqsubseteq$. Obviously, it is easy to obtain $((g_1, f_1)^\sqsubseteq \cup (g_2, f_2)^\sqsubseteq) = (g_1^A \cap f_1^1)^\sqsubseteq \cup (g_2^A \cap f_2^1)^\sqsubseteq$. Since $g_1 \leq g_1 \cup g_2$ and $f_2 \leq g_1 \cup g_2$, from Proposition 1, we have $(g_1 \cup g_2)^A \subseteq g_1^A$ and $(g_1 \cup g_2)^1 \subseteq f_1^1$. Furthermore, $(g_1 \cup g_2)^A \subseteq g_1^A \cup g_2^A \subseteq g_1^A \cap f_1^1 \cup g_2^A \cap f_2^1$. Likewise, since $f_1 \leq f_1 \cup f_2$ and $f_2 \leq f_1 \cup f_2$, from Proposition 2, we have $f_1^1 \subseteq (f_1 \cup f_2)^1$ and $f_2^1 \subseteq (f_1 \cup f_2)^1$. Furthermore, we have $f_1^1 \cup f_2^1 \subseteq (f_1 \cup f_2)^1$. To sum up, we can know $((g_1, f_1) \cup (g_2, f_2))^\sqsubseteq \subseteq (g_1^A \cup g_2^A)^A \cap (f_1^1 \cup f_2^1)^1 = (g_1, f_1)^\sqsubseteq \cup (g_2, f_2)^\sqsubseteq$. That is, $((g_1, f_1) \cup (g_2, f_2))^\sqsubseteq = (g_1, f_1)^\sqsubseteq \cup (g_2, f_2)^\sqsubseteq$. 

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(OS6) According to Definition 5, we have $(X_1 \cap X_2)^\cap = ((X_1 \cap X_2)^V, (X_1 \cap X_2)^I)$. $X_1^\cap = (X_1^V, X_1^I)$ and $X_2^\cap = (X_2^V, X_2^I)$. Obviously, it is easy to obtain that $X_1^\cap \cap X_2^\cap = (X_1^V \cap X_2^V, X_1^I \cap X_2^I)$. Due to $X_1 \cap X_2 \subseteq X_1$ and $X_1 \cap X_2 \subseteq X_2$, from Proposition 1, we know $X_1^\cap \subseteq (X_1 \cap X_2)^V$ and $X_2^\cap \subseteq (X_1 \cap X_2)^V$. Furthermore, we have $X_1^\cap \cap X_2^\cap \subseteq (X_1 \cap X_2)^V$. Likewise, from Proposition 2, we know $(X_1 \cap X_2)^I \subseteq X_1^I$ and $(X_1 \cap X_2)^I \subseteq X_2^I$. Therefore we have $(X_1 \cap X_2)^{\cap \cap} \subseteq X_1^{\cap \cap}$ and $(X_1 \cap X_2)^{\cap \cap} \subseteq X_2^{\cap \cap}$. To sum up, we can know $(X_1 \cap X_2)^{\cap \cap} = ((X_1 \cap X_2)^V, (X_1 \cap X_2)^I) \subseteq (X_1^V \cap X_2^V, X_1^I \cap X_2^I) = X_1^\cap \cap X_2^\cap$. That is, $(X_1 \cap X_2)^{\cap \cap} \subseteq X_1^{\cap \cap} \cap X_2^{\cap \cap}$.

Analogously, by Definition 5, we have $((g_1, f_1) \cap (g_2, f_2))^\cap = (g_1 \cap g_2, f_1 \cap f_2)^\cap = (g_1 \cap g_2)^A \cap (f_1 \cap f_2)^I$, $(g_1, f_1)^\cap = g_1^A \cap f_1^I$, and $(g_2, f_2)^\cap = g_2^A \cap f_2^I$. Obviously, it is easy to obtain that $(g_1, f_1)^\cap \cap (g_2, f_2)^\cap = (g_1 \cap g_2)^A \cap (f_1 \cap f_2)^I$. Note that from Proposition 1 and Proposition 2, we have $(g_1 \cap g_2)\Delta \ni g_1 \cap g_2 \ni f_1 \cap f_2$. To sum up, we get $((g_1, f_1) \cap (g_2, f_2))^\cap = (g_1 \cap g_2)^A \cap (f_1 \cap f_2)^I \ni (g_1, f_1)^\cap \cap (g_2, f_2)^\cap$. That is, $((g_1, f_1) \cap (g_2, f_2))^\cap \subseteq (g_1, f_1)^\cap \cap (g_2, f_2)^\cap$.

Definition 6: Let $(U, A \cup B, L, R)$ be a generalized one-sided formal context. A pair $(X, (g, f))$ is called an one sided three-way concept (OS-concept) if and only if

$X^\cap = (g, f)$ and $(g, f)^{\cap} = X$.

Where, $X \subseteq U, g \in \Pi_{a \in A} L_1 (a)$, and $f \in \Pi_{b \in B} L_2 (b)$. $X$ is the extension and $(g, f)$ is the intension of the OS-concept $(X, (g, f))$, respectively. In particular, $\forall x \in U, (x^{\cap}, x^{\cap})$ is a special OS-concept. All OS-concepts form a complete lattice (see Theorem 1), which is denoted by $OSL(U, A \cup B, L, R)$. That is, $OSL(U, A \cup B, L, R) = \{(X, (g, f)) | X^\cap = (g, f) \cap (g, f)^{\cap} = X\}$, which is also called One-Sided induced three-way concept lattice.

Analogously, by Definition 6, we have

$\forall (X_1, (g_1, f_1)), (X_2, (g_2, f_2)) \in OSL(U, A \cup B, L, R)$, the partial order is defined by $(X_1, (g_1, f_1)) \ni (X_2, (g_2, f_2)) \iff X_1 \ni X_2$ $(g_1, f_1) \ni (g_2, f_2))$.

Remark 3: For simplicity, throughout this paper we write $(x_1, x_2, x_3)$ instead of $(x_1, x_2, x_3)$ and (211) instead of (2,1,1).

Theorem 1: Let $\mathbb{K} = (U, A \cup B, L, R)$ be a generalized one-sided formal context. Then, $OSL(U, A \cup B, L, R)$ forms a complete lattice, i.e., $\forall (X, (g, f)) \ni OSL(U, A \cup B, L, R)$, the following two equations hold

$(1) \forall (i)\in i, (X, (g, f)) \ni OSL(U, A \cup B, L, R)$

$(2) \forall (i)\in i, (X, (g, f)) \ni OSL(U, A \cup B, L, R)$

Proof:

(1) In order to prove Equation (1), from Definition 6, we only need to prove

$(V_{i\in I} g_i, \Lambda_{i\in I} f_i)^{\cap \cap} = \cap_{i\in I} X_i$ and $(\Lambda_{i\in I} X_i)^{\cap \cap} = (V_{i\in I} g_i, \Lambda_{i\in I} f_i)^{\cap \cap}$.

Since, $(X_i, (g_i, f_i)) \in OSL(U, A \cup B, L, R)$ we have $X_i^\cap = g_i, X_i^I = f_i$ and $g_i \cap f_i = X_i$. On the one hand, by Proposition 1 and Proposition 2, we have $(V_{i\in I} g_i)^A = \Lambda_{i\in I} g_i^A, (\Lambda_{i\in I} f_i)^I = \Lambda_{i\in I} f_i^I$. Therefore we know $(V_{i\in I} g_i, \Lambda_{i\in I} f_i)^{\cap \cap} = (V_{i\in I} g_i, \Lambda_{i\in I} f_i)^{\cap \cap} = (V_{i\in I} g_i, \Lambda_{i\in I} f_i)^{\cap \cap}$.

(2) In order to prove Equation (2), we only need to prove

$(U_{i\in I} X_i)^{\cap \cap} = (\Lambda_{i\in I} g_i, V_{i\in I} f_i)$ and $(\Lambda_{i\in I} g_i, V_{i\in I} f_i)^{\cap \cap} = (U_{i\in I} X_i)^{\cap \cap}$.

Since $(X_i, (g_i, f_i)) \in OSL(U, A \cup B, L, R)$, by Definition 5 we have $X_i^\cap = g_i, X_i^I = f_i$ and $g_i \cap f_i = X_i$. 

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On the one hand, from Proposition 1, 2 and 3, we have 
\((U_{\in E}X_i)^{\omega} = (U_{\in E}X_i)^{\gamma} = \left(\bigcup_{i \in E} X_i\right)^{\gamma}, (U_{\in E}X_i)^{\gamma} = \bigcap_{i \in E} X_i^{\gamma} = \lambda_{i \in E} g_i^{\gamma}, (U_{\in E}X_i)^{\gamma} = \bigvee_{i \in E} f_i^{\gamma}.\) Thus, we get \((U_{\in E}X_i)^{\omega} = \left(\bigcup_{i \in E} X_i\right)^{\gamma}, (U_{\in E}X_i)^{\gamma} = \bigcap_{i \in E} X_i^{\gamma} = \lambda_{i \in E} g_i^{\gamma}, (U_{\in E}X_i)^{\gamma} = \bigvee_{i \in E} f_i^{\gamma}.\) On the other hand, from Definition 5, we have 
\((U_{\in E}X_i)^{\omega} = \left(\bigcup_{i \in E} X_i\right)^{\gamma} \cap \left(\bigcup_{i \in E} X_i\right)^{\gamma} = \left(\bigcap_{i \in E} X_i^{\gamma}\right)^{\Delta} \cap \left(\bigcup_{i \in E} X_i^{\gamma}\right)^{\Delta} = \left(\lambda_{i \in E} g_i^{\gamma}\right) \cap \left(\bigvee_{i \in E} f_i^{\gamma}\right)^{\gamma}.\) Combine with the above two cases, therefore we have 
\((U_{\in E}X_i)^{\omega} = \left(\bigcup_{i \in E} X_i\right)^{\gamma}, (U_{\in E}X_i)^{\gamma} = \bigcap_{i \in E} X_i^{\gamma} = \lambda_{i \in E} g_i^{\gamma}, (U_{\in E}X_i)^{\gamma} = \bigvee_{i \in E} f_i^{\gamma}.\)

(3) The rest is to prove that \(\lambda_{i \in E}(X_i, (g_i, f_i))\) and \(\bigvee_{i \in E}(X_i, (g_i, f_i))\) are the infimum and supremum of \(\{(X_i, (g_i, f_i))\mid i \in I\}\), respectively.

On the one hand, due to \(X_i \supseteq \bigcap_{i \in E} X_i\), we have \((X_i, (g_i, f_i)) \supseteq \left(\bigcap_{i \in E} X_i^{\gamma}\right)^{\Delta} \cap \left(\bigcup_{i \in E} X_i^{\gamma}\right)^{\Delta} = \left(\lambda_{i \in E} g_i^{\gamma}\right) \cap \left(\bigvee_{i \in E} f_i^{\gamma}\right)^{\gamma}.\) Thus, \(\lambda_{i \in E}(X_i, (g_i, f_i))\) is a lower bound of \(\{(X_i, (g_i, f_i))\mid i \in I\}\). On the other hand, if \((X, (g, f))\) is a lower bound of \(\{(X_i, (g_i, f_i))\mid i \in I\}\), then \(X \subseteq X_i\). It follows that \(X \subseteq \bigcap_{i \in E} X_i\). That is, \((X, (g, f)) \leq \lambda_{i \in E}(X_i, (g_i, f_i))\). From the above arguments, we get \(\lambda_{i \in E}(X_i, (g_i, f_i))\) is the infimum of \(\{(X_i, (g_i, f_i))\mid i \in I\}\).

Likewise, we can obtain \(\bigvee_{i \in E}(X_i, (g_i, f_i))\) is the supremum of \(\{(X_i, (g_i, f_i))\mid i \in I\}\).

Based on the algorithm of obtaining one-sided concept lattice in Butka and Pócs (2013), we propose one algorithm denoted by Algorithm 1 to compute all generalized one-sided three-way concepts.

**Algorithm 1**

Let \((U, A \cup B, L, R)\) be a generalized one-sided formal context. Let \(0_L\) denote the smallest element of \(\prod_{a \in A} L_1(a)\), and \(1_L\) denote the greatest element of \(\prod_{b \in B} L_2(b)\). Specific algorithms are as follows:

**Algorithm 1 Computing the all Generalized One-Sided Three-Way Concepts**

**Input:** a generalized formal context \((U, A \cup B, L, R)\);

**Output:** the set of all one-sided three-way concepts \(OSL(U, A \cup B, L, R)\);

1. Create lattice \(L = \prod_{a \in A} L_1(a) \cup \prod_{b \in B} L_2(b)\);
2. Initialize \(C \leftarrow \{(0_L, 1_L)\} ;\) /C is the set of intents
3. if \(U \neq \emptyset\)
4. then select \(x \in U\);
5. \(D = C\);
6. for each \((g, f) \in D\) do
7. \(C \leftarrow C \cup \{(g \land R(x, a), f \lor R(x, b))\} ;\)
8. end for
9. \(U \leftarrow U \setminus \{x\}\);
10. end if
11. for each \((g, f) \in C\) do
12. \(OSL(U, A \cup B, L, R) = OSL(U, A \cup B, L, R) \cup \{(g, f) ;\}
13. end for
14. return \(OSL(U, A \cup B, L, R)\).

**Example 3:** Let \((U, A \cup B, L, R)\) be a generalized one-sided formal context shown in Table 3. By algorithm 1, all OS-concepts induced from Table 3 can be computed as follows:

step 1. \(C = \{(0_L, 1_L)\} = \{(321,1111)\}\)
step 2. Select \(x_1 \in U\), then, \(C_1 = \{(321 \land R(x_1, a),1111 \lor R(x_1, b))\} = \{(201,2111)\}\), update \(C = C \cup C_1 = \{(321,1111),(201,2111)\}\)
step 3. Select \(x_2 \in U\) then \(C_2 = \{(321 \land R(x_2, a),1111 \lor R(x_2, b))\} = \{(311,3223)\}\)
update \(C = C \cup C_2 = \{(321,1111),(201,2111),(311,3223),(201,3232)\}\)
step 4. Select \(x_3 \in U\) then \(C_3 = \{(321 \land R(x_3, a),1111 \lor R(x_3, b))\} = \{(110,1212)\}\)
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\[ C_5 = \{(201 \land R(x_3,a), 2111 \lor R(x_3,b))\} = \{(100,2212)\} \]
\[ C_6 = \{(311 \land R(x_3,a), 3223 \lor R(x_3,b))\} = \{(110,3223)\} \]
\[ C_7 = \{(201 \land R(x_3,a), 3223 \lor R(x_3,b))\} = \{(100,3223)\} \]

update \[ C = C \cup C_4 \cup C_5 \cup C_6 \cup C_7 = \{(321,1111), (201,2111), (311,3223), (201,3223), (110,1212), (100,2212), (110,3223), (100,3223)\} \]

step 5. Select \( x_4 \in U \) then

\[ C_8 = \{(321 \land R(x_4,a), 1111 \lor R(x_4,b))\} = \{(321,3323)\} \]
\[ C_9 = \{(201 \land R(x_4,a), 2111 \lor R(x_4,b))\} = \{(201,3323)\} \]
\[ C_{10} = \{(311 \land R(x_4,a), 3223 \lor R(x_4,b))\} = \{(311,3323)\} \]
\[ C_{11} = \{(201 \land R(x_4,a), 3223 \lor R(x_4,b))\} = \{(201,3323)\} \]
\[ C_{12} = \{(110 \land R(x_4,a), 1212 \lor R(x_4,b))\} = \{(110,3323)\} \]
\[ C_{13} = \{(100 \land R(x_4,a), 2212 \lor R(x_4,b))\} = \{(100,3323)\} \]
\[ C_{14} = \{(110 \land R(x_4,a), 3223 \lor R(x_4,b))\} = \{(110,3323)\} \]
\[ C_{15} = \{(100 \land R(x_4,a), 3223 \lor R(x_4,b))\} = \{(100,3323)\} \]

update \[ C = C \cup C_4 \cup C_5 \cup C_6 \cup C_7 = \{(321,1111), (201,2111), (311,3223), (201,3223), (110,1212), (100,2212), (110,3223), (100,3223)\} \]

step 6. for each \( (g,f) \in C \) do

step 7. \( OSL(U,A \cup B, \mathcal{L}, R) = OSL(U,A \cup B, \mathcal{L}, R) \cup \{(g,f)^{\circ}, (g,f)\} \)

All OS-concepts are listed in Table 4, and the corresponding Hasse diagram of \( OSL(U,A \cup B, \mathcal{L}, R) \) is shown in Figure 1.

Table 4: All OS-Concepts of \( OSL(U,A \cup B, \mathcal{L}, R) \)

| OS1 \((x_1,(201,2111))\) | OS2 \((x_2,(311,3223))\) |
| OS3 \((x_3,(110,1212))\) | OS4 \((x_4,(321,3323))\) |
| OS5 \((x_1x_2,(201,3223))\) | OS6 \((x_1x_3,(100,2212))\) |
| OS7 \((x_2x_3,(110,3223))\) | OS8 \((x_2x_4,(311,3323))\) |
| OS9 \((x_1x_2x_3,(100,3223))\) | OS10 \((x_1x_2x_4,(201,3323))\) |
| OS11 \((x_2x_3x_4,(110,3323))\) | OS12 \((\emptyset,(321,1111))\) |
| OS13 \((x_1x_2x_3x_4,(100,3323))\) |

![Hasse diagram](http://www.cibtech.org/jpms.htm)

**Conclusion**

In this paper, as for a special type of generalized one-sided formal context, we have introduced one-sided...
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three-way operators (OS-operators) and one-sided three-way concepts (OS-concepts), which are based on positive attributes, negative attributes, positive operators and negative operators. Properties of OS-operators and OS-concepts have been discussed in detail. One algorithm has been presented to obtain all OS-concepts in a generalized one-sided formal context. Because many-value context is common in real life, we will focus on applications of many-value formal context in the future.

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REFERENCES


