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Review Article

NEW GEOMETRICAL PROPERTIES OF HYPERBOLA

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ABSTRACT
Hyperbola is one of the conic sections. It is the locus of a point that moves in such a way that the ratio of its distance from a fixed point (called Focus) to its distance from a fixed line (called Directrix) equals to constant 'e' which is greater than unity. The hyperbola is also very important in geometry and the field of Astronomy, since few comets are orbiting its star in a hyperbolical path and its star is as one of the foci. The objective of the research article is to establish few new theorems for mathematical properties related to various parameters of the hyperbola. These eight new properties have been described with necessary derivations of equations and appropriate drawings. The mathematical expressions of each theorem have also been defined. These theorems may be very useful to the research scholars for reference to the higher level research works.

Keywords: *Hyperbola, Conic Sections, Eccentricity of Hyperbola, Foci, Directrix, Transverse Axis, Conjugate Axis and Asymptotes of Hyperbola*

INTRODUCTION
A *hyperbola* (Weisstein Eric, 2003) is the mathematical shape that you obtain when vertically cutting a double cone. Many fields use hyperbolas in their designs and predictions of phenomena. In mathematics a hyperbola is specifically a smooth curve that lies in a plane. A hyperbola has two pieces, called connected components or branches, which are mirror images of each other and resembling two infinite bows. The hyperbola is one of the four kinds of conic section, formed by the intersection of a plane and a cone. Which conic section is formed depends on the angle the plane makes with the axis of the cone, compared with the angle a line on the surface of the cone makes with the axis of the *cone* (Weisstein Eric, 2003). If the angle between the plane and the axis is less than the angle between the line on the cone and the axis, or if the plane is parallel to the axis, then, the conic is a hyperbola.
Hyperbolas arise in practice in many ways: as the curve representing the function $f(x) = 1 / x$ in the Cartesian plane, as the appearance of a circle viewed from within it, as the path followed by the shadow of the tip of a sundial, as the shape of an open orbit, such as the orbit of a spacecraft during a gravity assisted swing by of a planet or more generally any spacecraft exceeding the escape velocity of the nearest planet, as the path of a single-apparition comet (one travelling too fast to ever return to the solar system), as the scattering trajectory of a subatomic particle (acted on by repulsive instead of attractive forces but the principle is the same), and so on.
Each branch of the hyperbola consists of two arms which become nearest to straighter further out from the center of the hyperbola. Diagonally opposite arms one from each branch tend in the limit to a common line, called the asymptote of those two arms. There are therefore two asymptotes, whose intersection is at the center of symmetry of the hyperbola, which can be thought of as the mirror point about which each branch reflects to form the other branch. In the case of the curve $f(x) = 1 / x$ the asymptotes are the two coordinate axes.
Hyperbolas share many of the ellipse's analytical properties such as eccentricity, focus, and directrix (Weisstein Eric, 2003). Typically, the correspondence can be made with nothing more than a change of sign in some term. Many other mathematical objects have their origin in the hyperbola, such as *hyperbolic paraboloids* (saddle surfaces), hyperboloids ("wastebaskets"), hyperbolic geometry (Lobachevsky's celebrated non-Euclidean geometry), hyperbolic functions like sinh(x), cosh(x), tanh(x), etc., and gyrovector spaces (a non-Euclidean geometry used in both relativity and quantum mechanics).

The feature of the hyperbola is its *asymptotes* (Weisstein Eric, 2003). For most practical purposes, the hyperbola can be considered as the asymptote itself except in the neighborhood of the origin.

Applications of Hyperbola

The Satellite systems make heavy use of hyperbolas and hyperbolic functions. When scientists launch a satellite into space, they must first use mathematical equations to predict its path. Because of the gravity influences of objects with heavy mass, the path of the satellite is skewed even though it may initially launch in a straight path. Using hyperbolas, astronomers can predict the path of the satellite to make adjustments so that the satellite gets to its destination.

The Radio system's signals employ hyperbolic functions. One important radio system, *LORAN* (Weisstein Eric, 2003) identified geographic positions using hyperbolas. Scientists and engineers established radio stations in positions according to the shape of a hyperbola in order to optimize the area covered by the signals from a station. LORAN allows people to locate objects over a wide area and played an important role in World War II.

The hyperbola has an important mathematical equation associated with it - the inverse relation. When an increase in one trait leads to a decrease in another or vice versa, the relationship can be described by a hyperbola. Graphing a hyperbola shows this immediately: when the x-value is small, the y-value is large, and vice versa. Many real-life situations can be described by the hyperbola, including the relationship between the pressure and volume of a gas.

Objects designed for use with our eyes make heavy use of hyperbolas. These objects include microscopes, telescopes and televisions. Before you can see a clear image of something, you need to focus on it. Your eyes have a natural focus point that does not allow you to see things too far away or close up. To view such things as planets or bacteria, scientists have designed objects that focus light into a single point. The designs of these use hyperbolas to reflect light to the focal point. When using a telescope or microscope, you are placing your eye in a well-planned focal point that allows the light from unseen objects to be focused in a way for you to view them.

Figure 1: Hyperboloid Natural Cooling Towers (http://www.piping-engineering.com/cooling-tower-heat-transfer-equipment-in-process-industry.html)

The light is directed towards one focus, but the hyperbolic reflector deflects the light to the other focus. This property can be useful in collecting light from stars. If a set of stars are roughly equidistant from the Earth, a hyperbolic reflector can reflect light rays from these stars to one of its foci.

A hyperbolic mirror can be used to take *panoramic photographs* (Weisstein Eric, 2003). A camera is pointed toward the vertex of the mirror and is positioned so that the lens is at one focus of the mirror.

They are encountered in the study of comets; the loran system of navigation for pleasure boats, ships, and Aircraft; sundials; capillary action; nuclear cooling towers; optical and radio telescopes; and contemporary architectural structures. The RCC cooling towers in Thermal power station of "NLC India Ltd." located at Neyveli, Cuddalore Dist., Tamil Nadu in India is a *hyperboloid* (Figure 1). With such structures, thin concrete shells can span large spaces. Some comets from outer space occasionally enter the sun's gravitational field, follow a hyperbolic path around the sun (with the sun as a focus), and then leave, never to be seen again.

Property-1 (Figure 2)

In a hyperbola if 'P' is one of the points on hyperbola, 'O' is the origin and points 'F' and 'G' are the foci then
$PF^2 + PG^2 = 2(OF^2 + OP^2)$

Derivation

Let, O is the origin, Co-ordinates of P be (x_1, y_1), co-ordinates of O be (0,0), Tangent is drawn at P(x_1, y_1). Points F & G are the foci of the hyperbola. Let, T is the meeting point of tangent and transverse axis and R is the meeting point of tangent and conjugate axis (Figure 2).

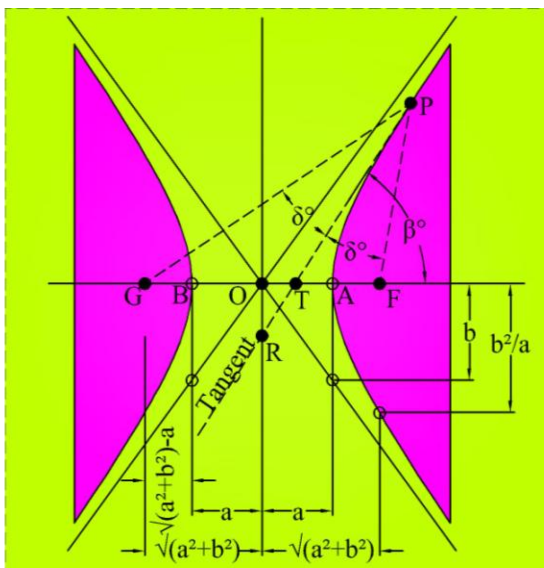

Figure 2: Typical Drawing of Hyperbola with Focal Length

We know that equation of tangent of a hyperbola is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ ——————————[1.1]

At point T, $y = 0$ and substituting in equation [1.1]

$$\frac{xx_1}{a^2} - \frac{0}{b^2} = 1$$

$$\therefore \frac{xx_1}{a^2} = 1$$

$$\therefore x = \frac{a^2}{x_1}$$

$\therefore$ coordinates of T is $\left(\frac{a^2}{x_1}, 0\right)$

At point R, $y = 0$ and substituting in equation [1.1]

$$\frac{0}{a^2} - \frac{yy_1}{b^2} = 1$$

$\therefore \dfrac{-yy_1}{b^2} = 1$

$\therefore\ y = \dfrac{-b^2}{y_1}$

$\therefore$ coordinates of T is $\left(0, \dfrac{-b^2}{y_1}\right)$

S. No	Point	Coordinates	
		x	y
1	O	0	0
2	F	$\sqrt{a^2+b^2}$	0
3	G	$-\sqrt{a^2+b^2}$	0
4	P	x_1	y_1
5	T	a^2/x_1	0
6	R	0	$-b^2/y_1$

$OP^2 = (x_1 - 0)^2 + (y_1 - 0)^2$

$\therefore OP^2 = {x_1}^2 + {y_1}^2$ ---------------------------[1.2]

$\therefore OF^2 = a^2 + b^2$ ---------------------------[1.3]

$PF^2 = \left(\sqrt{a^2+b^2} - x_1\right)^2 + (0 - y_1)^2$

$\therefore PF^2 = a^2 + b^2 + {x_1}^2 - 2x_1\sqrt{a^2+b^2} + {y_1}^2$

$\therefore PF^2 = {x_1}^2 + {y_1}^2 + a^2 + b^2 - 2x_1\sqrt{a^2+b^2}$ ----------------------- [1.4]

$PG^2 = \left(-\sqrt{a^2+b^2} - x_1\right)^2 + (0 - y_1)^2$

$\therefore PG^2 = a^2 + b^2 + {x_1}^2 + 2x_1\sqrt{a^2+b^2} + {y_1}^2$

$\therefore PG^2 = {x_1}^2 + {y_1}^2 + a^2 + b^2 + 2x_1\sqrt{a^2+b^2}$ ----------------------- [1.5]

Adding [1.4] & [1.5]

$PF^2 + PG^2 = 2({x_1}^2 + {y_1}^2) + 2(a^2 + b^2)$

$\therefore PF^2 + PG^2 = 2[({x_1}^2 + {y_1}^2) + (a^2 + b^2)]$ ----------------------- [1.6]

Adding [1.2] & [1.3]

$OF^2 + OP^2 = ({x_1}^2 + {y_1}^2) + (a^2 + b^2)$ ----------------------- [1.7]

Substituting, [1.7] in [1.6]

$PF^2 + PG^2 = 2(OF^2 + OP^2)$ ---------------------------[1.8]

Property-2 (Figure 2)

> *In a hyperbola if 'P' is one of the points on hyperbola 'O' is the origin and 'T' is the meeting point of primary axis & tangent drawn through 'P' and 'a' is the semi-major axis then*
> $PT^2 = OP^2 + OT^2 - 2a^2$

$OP^2 = (x_1 - 0)^2 + (y_1 - 0)^2$

$\therefore OP^2 = {x_1}^2 + {y_1}^2$ ---------------------------------------[2.1]

$OT^2 = \left(\dfrac{a^2}{x_1} - 0\right)^2 + (0 - 0)^2$

$\therefore OT^2 = \dfrac{a^4}{{x_1}^2}$ ------------------------------------- [2.2]

$$PT^2 = \left(\frac{a^2}{x_1} - x_1\right)^2 + (0 - y_1)^2$$

$$\therefore PT^2 = \frac{a^4}{{x_1}^2} + {x_1}^2 - 2x_1\left(\frac{a^2}{x_1}\right) + {y_1}^2$$

$$\therefore PT^2 = ({x_1}^2 + {y_1}^2) + \frac{a^4}{{x_1}^2} - 2a^2 \qquad \text{-----------------------[2.3]}$$

Substituting [2.1] & [2.2] in above

$$PT^2 = OP^2 + OT^2 - 2a^2 \qquad \text{---------------------------[2.4]}$$

Property-3 (Figure 2)

If 'P' is one of the points on hyperbola, 'O' is the origin and 'T' is the intersection of primary axis and tangent drawn through 'P', points 'F' and 'G' are the foci then
$FT^2 + GT^2 = 2(OT^2 + OP^2)$

Derivation

$$FT^2 = \left(\frac{a^2}{x_1} - \sqrt{a^2 + b^2}\right)^2 + (0 - 0)^2$$

$$\therefore FT^2 = \frac{a^4}{{x_1}^2} + a^2 + b^2 - 2\sqrt{a^2 + b^2}\left(\frac{a^2}{x_1}\right) \qquad \text{-----------------------[3.1]}$$

$$GT^2 = \left(\frac{a^2}{x_1} + \sqrt{a^2 + b^2}\right)^2 + (0 - 0)^2$$

$$\therefore FT^2 = \frac{a^4}{{x_1}^2} + a^2 + b^2 + 2\sqrt{a^2 + b^2}\left(\frac{a^2}{x_1}\right) \qquad \text{-----------------------[3.2]}$$

Adding [3.1] & [3.2]

$$FT^2 + GT^2 = 2\left(\frac{a^4}{{x_1}^2}\right) + 2(a^2 + b^2)$$

$$\therefore FT^2 + GT^2 = 2(OT^2 + OP^2) \qquad \text{-------------------------[3.3]}$$

Property-4 (Figure 2)

If 'P' is one of the points on hyperbola, 'O' is the origin and 'T' is the intersection of transverse axis and tangent drawn through 'P', points 'F' and 'G' are the foci then
$GT^2 - FT^2 = 4(OF)(OT)$

Derivation

Subtracting [3.2] from [3.1]

$$GT^2 - FT^2 = 4\sqrt{a^2 + b^2}\left(\frac{a^2}{x_1}\right)$$

$$\therefore GT^2 - FT^2 = 4(OF)(OT) \qquad \text{---------------------------[4.1]}$$

Property-5 (Figure 2)

If 'P' is any one of the points on hyperbola, 'O' is the origin, 'R' is the intersection of conjugate axis & tangent drawn through 'P' and 'b' is the semi-minor axis then
$PR^2 = OP^2 + OR^2 - 2b^2$

Derivation

$$PR^2 = (x_1 - 0)^2 + \left(y_1 + \frac{b^2}{y_1}\right)^2$$

$$\therefore PR^2 = x_1{}^2 + y_1{}^2 + \frac{b^4}{y_1{}^2} + 2y_1\left(\frac{b^2}{y_1}\right)$$

$$\therefore PR^2 = (x_1{}^2 + y_1{}^2) + \frac{b^4}{y_1{}^2} + 2b^2 \quad ------------------------[5.1]$$

Substituting [1.2] in above

$$PR^2 = OP^2 + OR^2 - 2b^2 \quad ------------------------- [5.2]$$

Property-6 (Figure 2)

If 'P' is any one of the points on hyperbola, 'O' is the origin, 'R' is the intersection of conjugate axis and tangent drawn through 'P', 'T' is the intersection of transverse axis and tangent drawn through 'P' and 'a' & 'b' are the semi-major axis & semi-minor axis respectively then
$PR^2 - PT^2 = OR^2 - OT^2 + 2(a^2 - b^2)$

Derivation

Subtracting [2.4] from [5.2]

$$PR^2 - PT^2 = OP^2 + OR^2 - 2b^2 - OP^2 - OT^2 + 2a^2$$

$$\therefore PR^2 - PT^2 = OR^2 - OT^2 + 2a^2 - 2b^2$$

$$PR^2 - PT^2 = OR^2 - OT^2 + 2(a^2 - b^2) \quad ------------------------[6.1]$$

Property-7 (Figure2)

If 'P' is any one of the points on hyperbola, 'O' is the origin, 'T' is the intersection point of transverse axis & tangent drawn through 'P' and points 'F' and 'G' are the foci then

$$\frac{PG^2}{PF^2} = \frac{OT(OP^2 + OF^2) + 2a^2OF}{OT(OP^2 + OF^2) - 2a^2OF}$$

Derivation

Dividing [1.5] by [1.4]

$$\frac{PG^2}{PF^2} = \frac{x_1{}^2 + y_1{}^2 + a^2 + b^2 + 2x_1\sqrt{a^2 + b^2}}{x_1{}^2 + y_1{}^2 + a^2 + b^2 - 2x_1\sqrt{a^2 + b^2}}$$

$$\therefore \frac{PG^2}{PF^2} = \frac{(x_1{}^2 + y_1{}^2) + (a^2 + b^2) + 2x_1\sqrt{a^2 + b^2}}{(x_1{}^2 + y_1{}^2) + (a^2 + b^2) - 2x_1\sqrt{a^2 + b^2}}$$

Substituting [1.2] & [1.3] in above

$$\therefore \frac{PG^2}{PF^2} = \frac{OP^2 + OF^2 + 2x_1OF}{OP^2 + OF^2 - 2x_1OF}$$

Substituting [1.2] & [1.3] in above

$$\therefore \frac{PG^2}{PF^2} = \frac{OP^2 + OF^2 + 2\left(\frac{a^2}{OT}\right)OF}{OP^2 + OF^2 - 2\left(\frac{a^2}{OT}\right)OF}$$

$$\frac{PG^2}{PF^2} = \frac{OT(OP^2 + OF^2) + 2a^2OF}{OT(OP^2 + OF^2) - 2a^2OF} \quad ------------------------ [7.1]$$

Property-8 (Figure 3)

If the tangent line drawn from at any point on a Hyperbola which meets the transverse axis at 'T' and point 'O' is the origin, α° angle between the tangent and transverse axis, β° is the angle between transverse axis and the line joining of point 'P' & 'O' and 'a' is the semi-major axis *then*

$$\frac{\tan\beta^\circ - \tan\alpha^\circ}{\tan\beta^\circ} = \frac{OT^2}{a^2}$$

Derivation
Referring Figure 3. In right-triangle MOP

$$\tan\alpha^\circ = \frac{PM}{OM}$$

$$\therefore \tan\alpha^\circ = \frac{y_1}{x_1} \qquad \text{-----------------------------------}[8.1]$$

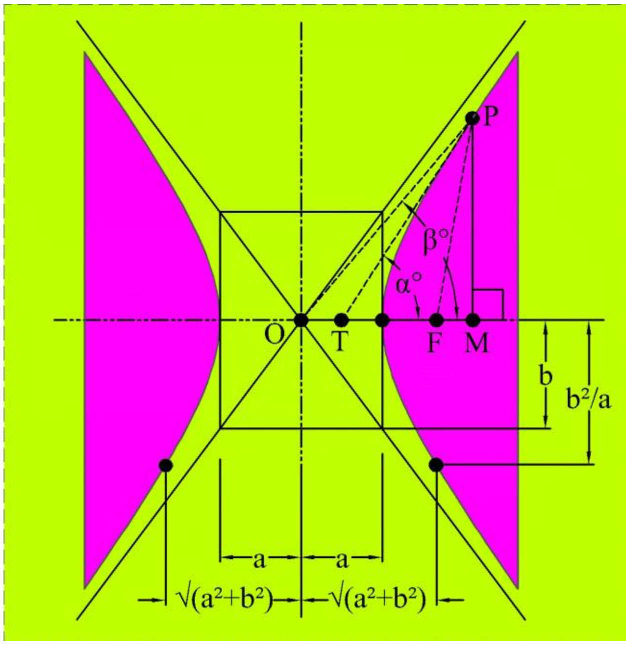

Figure 3: Typical Drawing of Hyperbola

In right-triangle MTP

$$\tan\beta^\circ = \frac{PM}{TM}$$

$$\therefore \tan\beta^\circ = \frac{y_1}{TM}$$

$$\therefore \tan\beta^\circ = \frac{y_1}{OM - OT}$$

$$\therefore \tan\beta^\circ = \frac{y_1}{x_1 - \left(\frac{a^2}{x_1}\right)}$$

$$\therefore \tan\beta^\circ = \frac{y_1}{\left(\frac{{x_1}^2 - a^2}{x_1}\right)}$$

$$\therefore \tan\beta^\circ = \frac{x_1 y_1}{{x_1}^2 - a^2}$$

From eqn. [8.1], $y_1 = x_1 \tan\alpha^\circ$ and substituting in above

$$\therefore \tan\beta^\circ = \frac{x_1(x_1 \tan\alpha^\circ)}{{x_1}^2 - a^2}$$

$$\therefore \tan\beta^\circ = \frac{{x_1}^2 \tan\alpha^\circ}{{x_1}^2 - a^2}$$

$$\therefore \tan\beta^\circ({x_1}^2 - a^2) = {x_1}^2 \tan\alpha^\circ \qquad \text{---------------------------}[8.2]$$

$$\therefore {x_1}^2 \tan\beta^\circ - a^2 \tan\beta^\circ = {x_1}^2 \tan\alpha^\circ$$

$$\therefore {x_1}^2 \tan\beta^\circ - {x_1}^2 \tan\alpha^\circ = a^2 \tan\beta^\circ$$

$$\therefore {x_1}^2(\tan\beta^\circ - \tan\alpha^\circ) = a^2 \tan\beta^\circ$$

$$\therefore \frac{\tan\beta^\circ - \tan\alpha^\circ}{\tan\beta^\circ} = \frac{a^2}{{x_1}^2}$$

$$\therefore \frac{\tan\beta^\circ - \tan\alpha^\circ}{\tan\beta^\circ} = \frac{OT^2}{a^2} \quad \text{------------------------} [8.3]$$

Conclusion

Hyperbola is one of the conic sections. Eight new properties have been developed by this Author and described with necessary derivations of equations and appropriate drawings. These properties, which have been defined in this paper is very useful for those doing research or further study in the field of conics or geometry, since these new properties also one of the important properties of an hyperbola.

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