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Review Article

BIANCHI TYPE V UNIVERSE FILLED WITHSCALAR FIELD COUPLED WITH ELECTROMAGNETIC FIELDIN $f(R,T)$ THEORY OF GRAVITY

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ABSTRACT
In $f(R,T)$ theory of gravity, we have studied the interacting scalar and electromagnetic fields in Bianchi type V space-time, by considering the general cases $f(R,T)=f_1(R)+\lambda f_2(T)$, $f(R,T)=f_1(R)f_2(T)$ and $f(R)$ theory and its particular cases $f(R,T)=R+\lambda T$, $f(R,T)=RT$ and $f(R)=R$. It is observed that, even though the line element of space-time are distinct, the convergent and isotropic solution of metric functions can be evolved in each case along with the components of vector potential, corresponding to suitable integrable function in particular cases.

Keywords*: Bianchi Type V, Scalar Field, Electromagnetic Field, $f(R,T)$ Theory of Gravity*

INTRODUCTION
Cosmological data from wide range of source have indicated that our universe is undergoing an accelerating expansion (Chandel and Shri Ram, 2015; Harko *et al.*, 2011; Myrzakulov, 2012; Noliri and Odintsov, 2003; Perlmutter *et al.*, 1998, 1999, 2003).
To explain this fact, two alternative theories are proposed: one concept of dark energy and other the amendment of general relativity leading to $f(R)$ and $f(R,T)$ theories (Perlmutter *et al.*, 1999; Riess *et al.*, 1998; Tonry *et al.*, 2003) where R stands for Ricci scalar $R=g^{ij}R_{ij}$, R_{ij} being Ricci tensor and T stands for trace of energy momentum tensor and $T=g^{ij}T_{ij}$,T_{ij} being energy momentum tensor derived from Lagrangian L_m. The field equations of $f(R,T)$ theories due to Harko *et al.*, (2011) are deduced by varying the action.

$$s=\int f(R,T)\sqrt{-g}d^4x+\int L_m\sqrt{-g}d^4x \qquad (1)$$

Where, L_m is lagrangian and the other symbols have their usual meaning. Energy momentum tensor is given by

$$T_{ij}=L_m g_{ij}-2\frac{\delta L_m}{\delta g^{ij}} \qquad (2)$$

Varying the action (1) with respect to g^{ij} which yields as

$$\delta s=\frac{1}{2\chi}\int\left\{f_R(R,T)\frac{\delta R}{\delta g^{ij}}+f_T(R,T)\frac{\delta T}{\delta g^{ij}}+\frac{f(R,T)}{\sqrt{-g}}\frac{\delta(\sqrt{-g})}{\delta g^{ij}}+\frac{2\chi}{\sqrt{-g}}\left(\frac{\delta(L_m\sqrt{-g})}{\delta g^{ij}}\right)\right\}\sqrt{-g}\,d^4x \qquad (3)$$

We define $\theta_{ij}=g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{ij}}$ (4)

By defining the generalized kronecker symbol $\frac{\delta g^{\alpha\beta}}{\delta g^{ij}}=\delta_i^\alpha\delta_j^\beta$ we can reduce

$$\frac{\delta g^{\alpha\beta}}{\delta g^{ij}}T_{\alpha\beta}=\delta_i^\alpha\delta_j^\beta T_{\alpha\beta}=g^{p\alpha}g_{pi}g^{q\beta}g_{qj}T_{\alpha\beta}=T_{ij}$$

Using above equations we can write

$$\frac{\delta T}{\delta g^{ij}}=\frac{\delta(g^{\alpha\beta}T_{\alpha\beta})}{\delta g^{ij}}=\frac{\delta g^{\alpha\beta}}{\delta g^{ij}}T_{\alpha\beta}+g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{ij}}=T_{ij}+\theta_{ij}$$

Integrating (3) we can obtain

$$f_R(R,T)R_{ij}-\frac{1}{2}f(R,T)g_{ij}+\left(g_{ij}\square-\nabla_i\nabla_j\right)f_R(R,T)=\chi T_{ij}-f_T(R,T)\left[T_{ij}+\theta_{ij}\right] \qquad (5)$$

This can be further rewritten as

$$f_R(R,T)G_{ij}+\frac{1}{2}[f_R(R,T)R-f(R,T)]g_{ij}+(g_{ij}\square-\nabla_i\nabla_j)f_R(R,T)=\chi T_{ij}-f_T(R,T)\left[T_{ij}+\theta_{ij}\right] \qquad (6)$$

Where, $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$

Taking trace of (5), we obtain

$$\square f_R(R,T) = \frac{2}{3}f(R,T) - \frac{1}{3}f_R(R,T)R + \frac{\chi}{3}T - \frac{1}{3}f_T(R,T)[T+\theta] \quad (7)$$

Using (7) the equation (6) can be organized in the form

$$G_j^i = \frac{1}{f_R(R,T)}\left[g^{mi}\nabla_m\nabla_j f_R(R,T)\right] - \frac{1}{6f_R(R,T)}[f_R(R,T)R + f(R,T)]g_j^i + \frac{\chi}{f_R(R,T)}\left[T_j^i - \frac{1}{3}Tg_j^i\right] + \frac{1}{3}\frac{f_T(R,T)}{f_R(R,T)}[T+\theta]g_j^i - \frac{f_T(R,T)}{f_R(R,T)}\left[T_j^i + \theta_j^i\right] \quad (8)$$

Let us now calculate the tensor θ_{ij}. Varying (2) with respect to metric tensor g^{ij} and using the definition (4) we obtain

$$\theta_{ij} = -T_{ij} + 2\left[\frac{\delta L_m}{\delta g^{ij}} - g^{\alpha\beta}\frac{\delta^2 L_m}{\delta g^{ij}\delta g^{\alpha\beta}}\right] \quad (9)$$

With this background, in this paper we discover the Bianchi type v space-time with interacting scalar field with electromagnetic one.

Matter Field Lagrangian L_m

The electromagnetic field tensor is given by

$$F_{ij} = \frac{\partial V_i}{\partial x^j} - \frac{\partial V_j}{\partial x^i},$$

Where, V_i is electromagnetic four potential.

The aforesaid the matter Lagrangian L_m can be expressed as

$$L_m = \left[\frac{1}{4}F_{\eta\tau}F^{\eta\tau} - \frac{1}{2}\varphi_{,\eta}\varphi^{,\eta}\psi\right] \quad (10)$$

Where, $\psi = \psi(I), \quad I = V_iV^i$

The function ψ characterizes the interaction between the scalar φ and electromagnetic field (Perlmutter *et al.*, 1999).

Then, the matter tensor in (2) can conveniently be expressed in the mixed form

$$T_j^i = \left(F_\alpha^i F_j^\alpha + \frac{1}{4}g_j^i F_{\alpha\beta}F^{\alpha\beta}\right) - \left[\frac{1}{2}\psi g_j^i - \dot{\psi}V^iV_j\right]\varphi_{,\eta}\varphi^{,\eta} + \psi\varphi^{,i}\varphi_{,j} \quad (11)$$

Similarly, (9) is written as

$$\theta_j^i = -T_j^i - \left(\psi - I\dot{\psi}\right)\varphi^{,i}\varphi_{,j} + I\ddot{\psi}\varphi_{,\eta}\varphi^{,\eta}V^iV_j \quad (12)$$

The equations (11) and (12), after contraction yield

$$T = -(\psi - I\dot{\psi})\varphi_{,\eta}\varphi^{,\eta} \quad (13)$$

$$\theta = I^2\ddot{\psi}\varphi_{,\eta}\varphi^{,\eta} \quad (14)$$

Bianchi Type V Space Time

The Bianchi type V metric of the space-time is specified by

$$ds^2 = dt^2 - A^2dx^2 - e^{2\alpha x}(B^2dy^2 + C^2dz^2), \quad (15)$$

Where, A, B and C are functions of cosmic time t and α is non-zero constant

For this metric the non-vanishing components of the Ricci tensors are

$$R_1^1 = \frac{-2\alpha^2}{A^2} + \frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC}, \quad R_2^2 = \frac{-2\alpha^2}{A^2} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC}, \quad R_3^3 = \frac{-2\alpha^2}{A^2} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC},$$

$$R_4^4 = \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}, \quad R_4^1 = \frac{2\alpha\dot{A}}{A^3} - \frac{\alpha\dot{B}}{A^2B} - \frac{\alpha\dot{C}}{A^2C}, \quad R_1^4 = -\frac{2\alpha\dot{A}}{A^3} + \frac{\alpha\dot{B}}{A^2B} + \frac{\alpha\dot{C}}{A^2C},$$

and the non-vanishing components of the Einstein tensors are

$$G_1^1 = \frac{\alpha^2}{A^2} - \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} - \frac{\dot{B}\dot{C}}{BC}, \quad G_2^2 = \frac{\alpha^2}{A^2} - \frac{\ddot{A}}{A} - \frac{\ddot{C}}{C} - \frac{\dot{A}\dot{C}}{AC}, \quad G_3^3 = \frac{\alpha^2}{A^2} - \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB},$$

$$G_4^4 = \frac{3\alpha^2}{A^2} - \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}\dot{C}}{BC} - \frac{\dot{A}\dot{C}}{AC}, \quad G_4^1 = \frac{2\alpha\dot{A}}{A^3} - \frac{\alpha\dot{B}}{A^2B} - \frac{\alpha\dot{C}}{A^2C}, \quad G_1^4 = -2\alpha\frac{\dot{A}}{A} + \alpha\frac{\dot{B}}{B} + \alpha\frac{\dot{C}}{C}$$

Electromagnetic Field Tensor F_{ij}

To achieve the compatibility with the non-static space time (15), we assume the electromagnetic vector potential in the form

$$\mathrm{V_i} = [\mathrm{u(x)V_1(t), V_2(t), V_3(t), V_4(t)}] \quad (16)$$

$$I = V_i V^i = -\left[\frac{u^2 {V_1}^2}{A^2} + \frac{{V_2}^2}{B^2}e^{-2\alpha x} + \frac{{V_3}^2}{C^2}e^{-2\alpha x} - {V_4}^2\right] \quad (17)$$

Then, it easy to deduce

$$F_{14} = u\dot{V}_1 \qquad F_{24} = \dot{V}_2 \qquad F_{34} = \dot{V}_3 \quad (18)$$

$$F_{ij}F^{ij} = -2\left[\frac{u^2 {\dot{V}_1}^2}{A^2} + \frac{{\dot{V}_2}^2}{B^2}e^{-2\alpha x} + \frac{{\dot{V}_3}^2}{C^2}e^{-2\alpha x}\right] \quad (19)$$

$$\varphi_i \varphi^i = \dot{\varphi}^2 \quad (20)$$

In reference to the above quantities at our disposal and space-time (15), the components of T_j^i in (11) assume the following values

$$T_1^1 = \frac{1}{2}u^2\frac{{\dot{V}_1}^2}{A^2} - \frac{1}{2}\frac{{\dot{V}_2}^2}{B^2}e^{-2\alpha x} - \frac{1}{2}\frac{{\dot{V}_3}^2}{C^2}e^{-2\alpha x} - \frac{1}{2}\psi\dot{\varphi}^2 - \dot{\psi}\dot{\varphi}^2 u^2\frac{{V_1}^2}{A^2} \quad (21a)$$

$$T_2^1 = \frac{u\dot{V}_1\dot{V}_2}{A^2} - \dot{\psi}\dot{\varphi}^2\frac{uV_1V_2}{A^2} \quad (21b)$$

$$T_3^1 = \frac{u\dot{V}_1\dot{V}_3}{A^2} - \dot{\psi}\dot{\varphi}^2\frac{uV_1V_3}{A^2} \quad (21c)$$

$$T_2^2 = -\frac{1}{2}u^2\frac{{\dot{V}_1}^2}{A^2} + \frac{1}{2}\frac{{\dot{V}_2}^2}{B^2}e^{-2\alpha x} - \frac{1}{2}\frac{{\dot{V}_3}^2}{C^2}e^{-2\alpha x} - \frac{1}{2}\psi\dot{\varphi}^2 - \dot{\psi}\dot{\varphi}^2\frac{{V_2}^2}{B^2}e^{-2\alpha x} \quad (21d)$$

$$T_3^2 = \frac{\dot{V}_2\dot{V}_3}{B^2}e^{-2\alpha x} - \dot{\psi}\dot{\varphi}^2\frac{V_2V_3}{B^2}e^{-2\alpha x} \quad (21e)$$

$$T_3^3 = -\frac{1}{2}u^2\frac{{\dot{V}_1}^2}{A^2} - \frac{1}{2}\frac{{\dot{V}_2}^2}{B^2}e^{-2\alpha x} + \frac{1}{2}\frac{{\dot{V}_3}^2}{C^2}e^{-2\alpha x} - \frac{1}{2}\psi\dot{\varphi}^2 - \dot{\psi}\dot{\varphi}^2\frac{{V_3}^2}{C^2}e^{-2\alpha x} \quad (21f)$$

$$T_4^4 = \frac{1}{2}u^2\frac{{\dot{V}_1}^2}{A^2} + \frac{1}{2}\frac{{\dot{V}_2}^2}{B^2}e^{-2\alpha x} + \frac{1}{2}\frac{{\dot{V}_3}^2}{C^2}e^{-2\alpha x} + \frac{1}{2}\psi\dot{\varphi}^2 + \dot{\psi}\dot{\varphi}^2{V_4}^2 \quad (21g)$$

$$T = -(\psi - I\dot{\psi})\dot{\varphi}^2 \quad (21h)$$

Similarly, the components of tensor θ_j^i from (12) can assume the following values

$$\theta_1^1 = -T_1^1 - I\ddot{\psi}\dot{\varphi}^2 u^2\frac{{V_1}^2}{A^2} \quad (22a)$$

$$\theta_2^1 = -T_2^1 - I\ddot{\psi}\dot{\varphi}^2 u\frac{V_1V_2}{A^2} \quad (22b)$$

$$\theta_3^1 = -T_3^1 - I\ddot{\psi}\dot{\varphi}^2 u\frac{V_1V_3}{A^2} \quad (22c)$$

$$\theta_2^2 = -T_2^2 - I\ddot{\psi}\dot{\varphi}^2\frac{{V_2}^2}{B^2}e^{-2\alpha x} \quad (22d)$$

$$\theta_3^2 = -T_3^2 - I\ddot{\psi}\dot{\varphi}^2\frac{V_2V_3}{B^2}e^{-2\alpha x} \quad (22e)$$

$$\theta_3^3 = -T_3^3 - I\ddot{\psi}\dot{\varphi}^2\frac{{V_3}^2}{C^2}e^{-2\alpha x} \quad (22f)$$

$$\theta_4^4 = -T_4^4 - \psi\dot{\varphi}^2 + I\dot{\psi}\dot{\varphi}^2 + I\ddot{\psi}\dot{\varphi}^2{V_4}^2 \quad (22g)$$

$$\theta = I^2\ddot{\psi}\dot{\varphi}^2 \quad (22h)$$

Variation of Lagrangian in (10) with respect to the electromagnetic field gives us

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^j}\left(\sqrt{-g}F^{ij}\right) - \left(\varphi_{,j}\varphi^{,j}\right)\dot{\psi}V^i = 0$$

$$\text{for } i = 1, j = 4 \Rightarrow \left(\frac{\dot{V}_1}{V_1}\right)^{\cdot} + \frac{{\dot{V}_1}^2}{{V_1}^2} + \frac{\dot{V}_1}{V_1}\left[\frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{\dot{A}}{A}\right] = \dot{\psi}\dot{\varphi}^2 \quad (23a)$$

$$\text{for } i = 2, j = 4 \Rightarrow \left(\frac{\dot{V}_2}{V_2}\right)^{\cdot} + \frac{{\dot{V}_2}^2}{{V_2}^2} + \frac{\dot{V}_2}{V_2}\left[\frac{\dot{C}}{C} + \frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right] = \dot{\psi}\dot{\varphi}^2 \quad (23b)$$

$$\text{for } i = 3, j = 4 \Rightarrow \left(\frac{\dot{V}_3}{V_3}\right)^{\cdot} + \frac{{\dot{V}_3}^2}{{V_3}^2} + \frac{\dot{V}_3}{V_3}\left[\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right] = \dot{\psi}\dot{\varphi}^2 \quad (23c)$$

$$\text{for } i = 4, j = 1 \Rightarrow u(x) = k_1 e^{-2\alpha x} \quad (23d)$$

$$\text{for } i = 4, j = 4 \Rightarrow V_4 = 0 \quad (23e)$$

where, k_1 is constant of integration.

Since the expression of the Einstein tensor in (8) is complicated, the solution of the Einstein's field equation in general cannot be obtained.

With this reality we take recourse to the particular cases of the function $f(R,T)$ and there upon try to obtain the solution.

Sub Case $\boldsymbol{f(R,T) = f_1(R) + \lambda f_2(T)}$

In this case we follow the notations $f_R(R,T) = \frac{\partial f(R,T)}{\partial R} = \dot{f}_1(R)$ $\quad f_T(R,T) = \frac{\partial f(R,T)}{\partial T} = \lambda \dot{f}_2(T)$

Then (8) reduces to the form

$$G_j^i = \frac{1}{\dot{f}_1(R)}\left[g^{mi}\nabla_m\nabla_j \dot{f}_1(R)\right] - \frac{1}{6\dot{f}_1(R)}\left[\dot{f}_1(R)R + f_1(R) + \lambda f_2(T)\right]g_j^i + \frac{\chi}{\dot{f}_1(R)}\left[T_j^i - \frac{1}{3}Tg_j^i\right] + \frac{\lambda}{3}\frac{\dot{f}_2(T)}{\dot{f}_1(R)}[T+\theta]g_j^i - \frac{\lambda \dot{f}_2(T)}{\dot{f}_1(R)}\left[T_j^i + \theta_j^i\right] \quad (24)$$

Since for the space-time (15) the components $G_2^1 = 0, G_3^1 = 0, G_3^2 = 0$ the field equation (24), with the help of (21) and (22), yield

$$\frac{\dot{V}_1\dot{V}_2}{V_1V_2} = \dot{\psi}\dot{\varphi}^2 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2 \quad (25a)$$

$$\frac{\dot{V}_1\dot{V}_3}{V_1V_3} = \dot{\psi}\dot{\varphi}^2 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2 \quad (25b)$$

$$\frac{\dot{V}_2\dot{V}_3}{V_2V_3} = \dot{\psi}\dot{\varphi}^2 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2 \quad (25c)$$

From (25) we can write

$$\frac{\dot{V}_1\dot{V}_2}{V_1V_2} = \frac{\dot{V}_2\dot{V}_3}{V_2V_3} = \frac{\dot{V}_1\dot{V}_3}{V_1V_3} = \dot{\psi}\dot{\varphi}^2 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2 \quad (26)$$

Further we can rewrite as

$$\frac{\dot{V}_1}{V_1} = \frac{\dot{V}_2}{V_2} = \frac{\dot{V}_3}{V_3} \equiv \frac{\dot{h}_1}{h_1}, \text{ say} \quad (27)$$

Where, h_1 is some unknown function of t

Inserting (27) in (26) we can have

$$\left(\frac{\dot{h}_1}{h_1}\right)^2 = \left(\frac{\dot{h}_1}{h_1}\right)^2 = \left(\frac{\dot{h}_1}{h_1}\right)^2 = \dot{\psi}\dot{\varphi}^2 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2 \quad (28)$$

Upon integration of the equation (27) we get

$$V_1 = k_2h_1 \qquad V_2 = k_3h_1 \qquad V_3 = k_4h_1 \quad (29)$$

where, k_2, k_3, k_4 are constants of integration.

Now our plan is to express the components of T_j^i in (21) in terms of T_4^4. For this we consider the expression

$$u^2\frac{\dot{V}_1^2}{A^2} + \frac{\dot{V}_2^2}{B^2}e^{-2\alpha x} + \frac{\dot{V}_3^2}{C^2}e^{-2\alpha x} = \left[\frac{u^2v_1^2}{A^2} + \frac{V_2^2}{B^2}e^{-2\alpha x} + \frac{V_3^2}{C^2}e^{-2\alpha x}\right]\left(\frac{\dot{h}_1}{h_1}\right)^2 \quad \text{by (27)}$$

$$= -I\left(\frac{\dot{h}_1}{h_1}\right)^2 \quad \text{by (17) and (23e)}$$

$$= \frac{\lambda}{\chi}\dot{f}_2(T)I^2\ddot{\psi}\dot{\varphi}^2 - I\dot{\psi}\dot{\varphi}^2 \quad \text{by (28)} \quad (30)$$

We attempt to express the components of T_j^i in (21) in terms of T_4^4 by using (27), (28) and (30)

$$T_4^4 = \frac{1}{2}\psi\dot{\varphi}^2 - \frac{1}{2}I\dot{\psi}\dot{\varphi}^2 + \frac{1}{2}\frac{\lambda}{\chi}\dot{f}_2(T)I^2\ddot{\psi}\dot{\varphi}^2 \quad (31a)$$

$$T_1^1 = -T_4^4 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2u^2\frac{V_1^2}{A^2} \quad (31b)$$

$$T_2^1 = -\frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2u\frac{V_1V_2}{A^2} \quad (31c)$$

$$T_3^1 = -\frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2u\frac{V_1V_3}{A^2} \quad (31d)$$

$$T_2^2 = -T_4^4 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2\frac{V_2^2}{B^2}e^{-2\alpha x} \quad (31e)$$

$$T_3^2 = -\frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2\frac{V_2V_3}{B^2}e^{-2\alpha x} \quad (31f)$$

$$T_3^3 = -T_4^4 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2\frac{V_3^2}{C^2}e^{-2\alpha x} \quad (31g)$$

$$T = -(\psi - I\dot{\psi}) \quad (31h)$$

Now we consider the non-vanishing components of Einstein tensor G_1^1, G_2^2, G_3^3 from (24)

$$\frac{\alpha^2}{A^2}-\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}-\frac{\dot{B}\dot{C}}{BC}=-\frac{\dot{A}}{A}\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}-\frac{1}{6\dot{f}_1(R)}\left[\dot{f}_1(R)R+f_1(R)+\lambda f_2(T)\right]+\frac{\chi}{\dot{f}_1(R)}\left[T_1^1-\frac{1}{3}T\right]+\frac{\lambda}{3}\frac{\dot{f}_2(T)}{\dot{f}_1(R)}[T+\theta]-\frac{\lambda\dot{f}_2(T)}{\dot{f}_1(R)}[T_1^1+\theta_1^1] \quad (32a)$$

$$\frac{\alpha^2}{A^2}-\frac{\ddot{C}}{C}-\frac{\ddot{A}}{A}-\frac{\dot{C}\dot{A}}{CA}=-\frac{\dot{B}}{B}\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}-\frac{1}{6\dot{f}_1(R)}\left[\dot{f}_1(R)R+f_1(R)+\lambda f_2(T)\right]+\frac{\chi}{\dot{f}_1(R)}\left[T_2^2-\frac{1}{3}T\right]+\frac{\lambda}{3}\frac{\dot{f}_2(T)}{\dot{f}_1(R)}[T+\theta]-\frac{\lambda\dot{f}_2(T)}{\dot{f}_1(R)}[T_2^2+\theta_2^2] \quad (32b)$$

$$\frac{\alpha^2}{A^2}-\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}-\frac{\dot{A}\dot{B}}{AB}=-\frac{\dot{C}}{C}\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}-\frac{1}{6\dot{f}_1(R)}\left[\dot{f}_1(R)R+f_1(R)+\lambda f_2(T)\right]+\frac{\chi}{\dot{f}_1(R)}\left[T_3^3-\frac{1}{3}T\right]+\frac{\lambda}{3}\frac{\dot{f}_2(T)}{\dot{f}_1(R)}[T+\theta]-\frac{\lambda\dot{f}_2(T)}{\dot{f}_1(R)}[T_3^3+\theta_3^3] \quad (32c)$$

Subtracting (32b) from (32a), (32c) from (32b) and (32a) from (32c) we get

$$\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}+\frac{\dot{C}}{C}\left[\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right]+\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}=\frac{\chi}{\dot{f}_1(R)}[T_1^1-T_2^2]+\frac{\lambda\dot{f}_2(T)}{\dot{f}_1(R)}[(T_2^2+\theta_2^2)-(T_1^1+\theta_1^1)] \quad (33a)$$

$$\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}+\frac{\dot{A}}{A}\left[\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right]+\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}=\frac{\chi}{\dot{f}_1(R)}[T_2^2-T_3^3]+\frac{\lambda\dot{f}_2(T)}{\dot{f}_1(R)}[(T_3^3+\theta_3^3)-(T_2^2+\theta_2^2)] \quad (33b)$$

$$\frac{\ddot{C}}{C}-\frac{\ddot{A}}{A}+\frac{\dot{B}}{B}\left[\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right]+\left(\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right)\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}=\frac{\chi}{\dot{f}_1(R)}[T_3^3-T_1^1]+\frac{\lambda\dot{f}_2(T)}{\dot{f}_1(R)}[(T_1^1+\theta_1^1)-(T_3^3+\theta_3^3)] \quad (33c)$$

With the help of (31) and (22) the equation (33) reduces to

$$\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}+\frac{\dot{C}}{C}\left[\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right]+\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}=0 \quad (34a)$$

$$\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}+\frac{\dot{A}}{A}\left[\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right]+\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}=0 \quad (34b)$$

$$\frac{\ddot{C}}{C}-\frac{\ddot{A}}{A}+\frac{\dot{B}}{B}\left[\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right]+\left(\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right)\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}=0 \quad (34c)$$

Up on the integration of equations (34) yield

$$\frac{A}{B}=k_6 exp\left\{k_5\int\frac{1}{ABC\dot{f}_1(R)}dt\right\} \quad (35a)$$

$$\frac{B}{C}=k_8 exp\left\{k_7\int\frac{1}{ABC\dot{f}_1(R)}dt\right\} \quad (35b)$$

$$\frac{C}{A}=k_{10} exp\left\{k_9\int\frac{1}{ABC\dot{f}_1(R)}dt\right\} \quad (35c)$$

Where, $k's$ are constants of integration with the condition that
$k_6k_8k_{10}=1$ and $k_5+k_7+k_9=0$

From (35), we can express the values of A, B, C explicitly as

$$A=(ABC)^{\frac{1}{3}}k_{11} exp\left\{k_{12}\int\frac{1}{ABC\dot{f}_1(R)}dt\right\} \quad (36a)$$

$$B=(ABC)^{\frac{1}{3}}k_{13} exp\left\{k_{14}\int\frac{1}{ABC\dot{f}_1(R)}dt\right\} \quad (36b)$$

$$C=(ABC)^{\frac{1}{3}}k_{15} exp\left\{k_{16}\int\frac{1}{ABC\dot{f}_1(R)}dt\right\} \quad (36c)$$

Where, $k's$ are constant of integration.

Using (27) the equation (23) can be written as

$$\left(\frac{\dot{h}_1}{h_1}\right)^{\cdot}+\frac{\dot{h}_1{}^2}{h_1{}^2}+\frac{\dot{h}_1}{h_1}\left[\frac{\dot{C}}{C}+\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right]=\dot{\psi}\dot{\varphi}^2 \quad (37a)$$

$$\left(\frac{\dot{h}_1}{h_1}\right)^{\cdot}+\frac{\dot{h}_1{}^2}{h_1{}^2}+\frac{\dot{h}_1}{h_1}\left[\frac{\dot{A}}{A}+\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right]=\dot{\psi}\dot{\varphi}^2 \quad (37b)$$

$$\left(\frac{\dot{h}_1}{h_1}\right)^{\cdot}+\frac{\dot{h}_1{}^2}{h_1{}^2}+\frac{\dot{h}_1}{h_1}\left[\frac{\dot{B}}{B}+\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right]=\dot{\psi}\dot{\varphi}^2 \quad (37c)$$

These equations further imply

$$\frac{\dot{C}}{C}+\frac{\dot{A}}{A}-\frac{\dot{B}}{B}=\frac{\dot{A}}{A}+\frac{\dot{B}}{B}-\frac{\dot{C}}{C}=\frac{\dot{B}}{B}+\frac{\dot{C}}{C}-\frac{\dot{A}}{A}$$

or $\frac{\dot{A}}{A}=\frac{\dot{B}}{B}=\frac{\dot{C}}{C}$ (38)

Inserting (38) in (37), we get

$$\left(\frac{\dot{h}_1}{h_1}\right)^{\cdot} + \frac{\dot{h}_1{}^2}{h_1{}^2} + \frac{\dot{h}_1}{h_1}\left[\frac{\dot{A}}{A}\right] = \dot{\psi}\dot{\varphi}^2 \qquad (39)$$

But from (28) $\dot{\psi}\dot{\varphi}^2 = \frac{\dot{h}_1{}^2}{h_1{}^2} + \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2$ (40)

Inserting (40) in (39), we get

$$\left(\frac{\dot{h}_1}{h_1}\right)^{\cdot} + \frac{\dot{h}_1}{h_1}\left[\frac{\dot{A}}{A}\right] = \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2 \qquad (41)$$

If we confine the function ψ as linear function of I i.e. $\ddot{\psi} = 0$ or $\psi = k_{11}I + k_{12}$ and solution of (41) is

$$h_1 = k_{14}exp\left\{k_{13}\int\frac{1}{A}dt\right\} \qquad (42)$$

Where, $k's$ are constants of integration.

With the help of (42) the equations in (29) convert in to

$$V_1 = k_{15}exp\left\{k_{13}\int\frac{1}{A}dt\right\} \qquad (43a)$$

$$V_2 = k_{16}exp\left\{k_{13}\int\frac{1}{A}dt\right\} \qquad (43b)$$

$$V_3 = k_{17}exp\left\{k_{13}\int\frac{1}{A}dt\right\} \qquad (43c)$$

Where, $k's$ are constants of integration.

Sub Case $\boldsymbol{f(R,T) = f_1(R)f_2(T)}$

In this case we follow the notations

$$f_R(R,T) = \frac{\partial f(R,T)}{\partial R} = \dot{f}_1(R)f_2(T)\,, \qquad f_T(R,T) = \frac{\partial f(R,T)}{\partial T} = f_1(R)\dot{f}_2(T)$$

Then, equation (8) reduces to

$$G_j^{\mu} = \frac{1}{\dot{f}_1(R)f_2(T)}\left[g^{\mu i}\nabla_i\nabla_j\dot{f}_1(R)f_2(T)\right] - \frac{1}{6\dot{f}_1(R)f_2(T)}\left[\dot{f}_1(R)f_2(T)R + f_1(R)f_2(T)\right]g_j^{\mu} + \frac{\chi}{\dot{f}_1(R)f_2(T)}\left[T_j^{\mu} - \frac{1}{3}Tg_j^{\mu}\right] + \frac{1}{3}\frac{f_1(R)\dot{f}_2(T)}{\dot{f}_1(R)f_2(T)}[T+\theta]g_j^{\mu} - \frac{f_1(R)\dot{f}_2(T)}{\dot{f}_1(R)f_2(T)}\left[T_j^{\mu} + \theta_j^{\mu}\right] \qquad (44)$$

Since, for the space-time (15), $G_2^1 = 0, G_3^1 = 0, G_3^2 = 0$ from (44) and by using (21) and (22)

$$\frac{\dot{V}_1\dot{V}_2}{V_1V_2} = \dot{\psi}\,\dot{\varphi}^2 - \frac{f_1(R)\dot{f}_2(T)}{\chi}I\ddot{\psi}\dot{\varphi}^2 \qquad (45a)$$

$$\frac{\dot{V}_1\dot{V}_3}{V_1V_3} = \dot{\psi}\,\dot{\varphi}^2 - \frac{f_1(R)\dot{f}_2(T)}{\chi}I\ddot{\psi}\dot{\varphi}^2 \qquad (45b)$$

$$\frac{\dot{V}_2\dot{V}_3}{V_2V_3} = \dot{\psi}\,\dot{\varphi}^2 - \frac{f_1(R)\dot{f}_2(T)}{\chi}I\ddot{\psi}\dot{\varphi}^2 \qquad (45c)$$

From (45), we can write

$$\frac{\dot{V}_1\dot{V}_2}{V_1V_2} = \frac{\dot{V}_2\dot{V}_3}{V_2V_3} = \frac{\dot{V}_1\dot{V}_3}{V_1V_3} = \dot{\psi}\,\dot{\varphi}^2 - \frac{f_1(R)\dot{f}_2(T)}{\chi}I\ddot{\psi}\dot{\varphi}^2 \qquad (46)$$

Further we can rewrite it as

$$\frac{\dot{V}_1}{V_1} = \frac{\dot{V}_2}{V_2} = \frac{\dot{V}_3}{V_3} \equiv \frac{\dot{h}_7}{h_7}\text{, say} \qquad (47)$$

Where, h_7 is some unknown function of t.

Inserting (47) in (46), yields

$$\left(\frac{\dot{h}_7}{h_7}\right)^2 = \left(\frac{\dot{h}_7}{h_7}\right)^2 = \left(\frac{\dot{h}_7}{h_7}\right)^2 = \dot{\psi}\,\dot{\varphi}^2 - \frac{f_1(R)\dot{f}_2(T)}{\chi}I\ddot{\psi}\dot{\varphi}^2 \qquad (48)$$

Integrating (47) with respect to t, we get

$$V_1 = m_1h_7 \quad V_2 = m_2h_7 \quad V_3 = m_3h_7 \qquad (49)$$

Where, m_1, m_2, m_3 are constants of integration.

Now our plan is to express the components of T_j^i in (3.7) in terms of T_4^4. For this we consider the expression

$$u^2\frac{\dot{V}_1{}^2}{A^2} + \frac{\dot{V}_2{}^2}{B^2}e^{-2\alpha x} + \frac{\dot{V}_3{}^2}{C^2}e^{-2\alpha x} = \left[\frac{u^2V_1{}^2}{A^2} + \frac{V_2{}^2}{B^2}e^{-2\alpha x} + \frac{V_3{}^2}{C^2}e^{-2\alpha x}\right]\left(\frac{\dot{h}_7}{h_7}\right)^2 \quad \text{by (47)}$$

$= -I\left(\frac{\dot{h}_7}{h_7}\right)^2$ by (17) and (23e)

$= \frac{f_1(R)\dot{f}_2(T)}{\chi} I^2\ddot{\psi}\dot{\varphi}^2 - I\dot{\psi}\dot{\varphi}^2$ by (48) (50)

We attempt to express all components of T_j^i in (21) in terms of T_4^4 by using (47), (48) and (50)

$$T_4^4 = \frac{1}{2}\psi\dot{\varphi}^2 - \frac{1}{2}I\dot{\psi}\dot{\varphi}^2 + \frac{1}{2}\frac{f_1(R)\dot{f}_2(T)}{\chi} I^2\ddot{\psi}\dot{\varphi}^2 \quad (51a)$$

$$T_1^1 = -T_4^4 - \frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2 \frac{u^2 {V_1}^2}{A^2} \quad (51b)$$

$$T_2^1 = -\frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2 \frac{uV_1V_2}{A^2} \quad (51c)$$

$$T_3^1 = -\frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2 \frac{uV_1V_3}{A^2} \quad (51d)$$

$$T_2^2 = -T_4^4 - \frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2 \frac{{V_2}^2}{B^2} e^{-2\alpha x} \quad (51e)$$

$$T_3^2 = -\frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2 \frac{V_2V_3}{B^2} e^{-2\alpha x} \quad (51f)$$

$$T_3^3 = -T_4^4 - \frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2 \frac{{V_3}^2}{C^2} e^{-2\alpha x} \quad (51g)$$

$$T = -(\psi - I\dot{\psi})\dot{\varphi}^2 \quad (51h)$$

We consider the non-vanishing components of Einstein tensor G_1^1, G_2^2, G_3^3 from (44), we can deduce

$$\frac{\alpha^2}{A^2} - \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} - \frac{\dot{B}\dot{C}}{BC} = -\frac{1}{A^2}\left[\frac{\ddot{f}_2(T)}{f_2(T)}\left(\frac{dT}{dx}\right)^2 + \frac{\dot{f}_2(T)}{f_2(T)}\frac{d^2T}{dx^2}\right] - \frac{\dot{A}}{A}\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt} + \frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right] - \frac{1}{6\dot{f}_1(R)f_2(T)} \times$$
$$\left[\dot{f}_1(R)f_2(T)\mathrm{R} + \mathrm{f}_1(\mathrm{R})\mathrm{f}_2(\mathrm{T})\right] + \frac{\chi}{\dot{f}_1(R)f_2(T)}\left[T_1^1 - \frac{1}{3}T\right] + \frac{1}{3}\frac{\mathrm{f}_1(\mathrm{R})\dot{\mathrm{f}}_2(\mathrm{T})}{\dot{f}_1(R)f_2(T)}[T+\theta] -$$
$$\frac{\mathrm{f}_1(\mathrm{R})\dot{\mathrm{f}}_2(\mathrm{T})}{\dot{f}_1(R)f_2(T)}[T_1^1 + \theta_1^1] \quad (52a)$$

$$\frac{\alpha^2}{A^2} - \frac{\ddot{C}}{C} - \frac{\ddot{A}}{A} - \frac{\dot{C}\dot{A}}{CA} = \frac{\alpha}{A^2}\left[\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dx}\right] - \frac{\dot{B}}{B}\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt} + \frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right] - \frac{1}{6\dot{f}_1(R)f_2(T)} \times$$
$$\left[\dot{f}_1(R)f_2(T)\mathrm{R} + \mathrm{f}_1(\mathrm{R})\mathrm{f}_2(\mathrm{T})\right] + \frac{\chi}{\dot{f}_1(R)f_2(T)}\left[T_2^2 - \frac{1}{3}T\right] + \frac{1}{3}\frac{\mathrm{f}_1(\mathrm{R})\dot{\mathrm{f}}_2(\mathrm{T})}{\dot{f}_1(R)f_2(T)}[T+\theta] -$$
$$\frac{\mathrm{f}_1(\mathrm{R})\dot{\mathrm{f}}_2(\mathrm{T})}{\dot{f}_1(R)f_2(T)}[T_2^2 + \theta_2^2] \quad (52b)$$

$$\frac{\alpha^2}{A^2} - \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} = \frac{\alpha}{A^2}\left[\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dx}\right] - \frac{\dot{C}}{C}\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt} + \frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right] - \frac{1}{6\dot{f}_1(R)f_2(T)} \times$$
$$\left[\dot{f}_1(R)f_2(T)\mathrm{R} + \mathrm{f}_1(\mathrm{R})\mathrm{f}_2(\mathrm{T})\right] + \frac{\chi}{\dot{f}_1(R)f_2(T)}\left[T_3^3 - \frac{1}{3}T\right] + \frac{1}{3}\frac{\mathrm{f}_1(\mathrm{R})\dot{\mathrm{f}}_2(\mathrm{T})}{\dot{f}_1(R)f_2(T)}[T+\theta] -$$
$$\frac{\mathrm{f}_1(\mathrm{R})\dot{\mathrm{f}}_2(\mathrm{T})}{\dot{f}_1(R)f_2(T)}[T_3^3 + \theta_3^3] \quad (52c)$$

Subtracting (52b) from (52a), (52c) from (52b) and (52a) from (52c) we obtain

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{C}}{C}\left[\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right] + \left[\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right]\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt} + \frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right] = -\frac{1}{A^2}\frac{\ddot{f}_2(T)}{f_2(T)}\left(\frac{dT}{dx}\right)^2 - \frac{1}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{d^2T}{dx^2} - \frac{\alpha}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dx}$$
$$\frac{\chi}{\dot{f}_1(R)f_2(T)}[T_1^1 - T_2^2] + \frac{\mathrm{f}_1(\mathrm{R})\dot{\mathrm{f}}_2(\mathrm{T})}{\dot{f}_1(R)f_2(T)}[(T_2^2 + \theta_2^2) - (T_1^1 + \theta_1^1)] \quad (53a)$$

$$\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} + \frac{\dot{\mathrm{A}}}{A}\left[\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right] + \left[\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right]\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt} + \frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right] = \frac{\chi}{\dot{f}_1(R)f_2(T)}[T_2^2 - T_3^3] + \frac{\mathrm{f}_1(\mathrm{R})\dot{\mathrm{f}}_2(\mathrm{T})}{\dot{f}_1(R)f_2(T)} \times$$
$$[(T_3^3 + \theta_3^3) - (T_2^2 + \theta_2^2)] \quad (53b)$$

$$\frac{\ddot{C}}{C} - \frac{\ddot{A}}{A} + \frac{\dot{\mathrm{B}}}{B}\left[\frac{\dot{C}}{C} - \frac{\dot{A}}{A}\right] + \left[\frac{\dot{C}}{C} - \frac{\dot{A}}{A}\right]\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt} + \frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right] = \frac{1}{A^2}\frac{\ddot{f}_2(T)}{f_2(T)}\left(\frac{dT}{dx}\right)^2 + \frac{1}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{d^2T}{dx^2} + \frac{\alpha}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dx}$$
$$+ \frac{\chi}{\dot{f}_1(R)f_2(T)}[T_3^3 - T_1^1] + \frac{\mathrm{f}_1(\mathrm{R})\dot{\mathrm{f}}_2(\mathrm{T})}{\dot{f}_1(R)f_2(T)}[(T_1^1 + \theta_1^1) - (T_3^3 + \theta_3^3)] \quad (53c)$$

By using (51) and (22) we obtain

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{C}}{C}\left[\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right] + \left[\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right]\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt} + \frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right] = -\frac{1}{A^2}\frac{\ddot{f}_2(T)}{f_2(T)}\left(\frac{dT}{dx}\right)^2 - \frac{1}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{d^2T}{dx^2} - \frac{\alpha}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dx} \quad (54a)$$

$$\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}+\frac{\dot{A}}{A}\left[\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right]+\left[\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right]\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right]=0 \quad (54b)$$

$$\frac{\ddot{C}}{C}-\frac{\ddot{A}}{A}+\frac{\dot{B}}{B}\left[\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right]+\left[\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right]\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right]=\frac{1}{A^2}\frac{\ddot{f}_2(T)}{f_2(T)}\left(\frac{dT}{dx}\right)^2+\frac{1}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{d^2T}{dx^2}+\frac{\alpha}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dx} \quad (54c)$$

If we take the relation between metric function A and C as

$$A=m_4C, \quad (55)$$

Where, m_4 is nonzero positive constant. Then, we obtain

$$\frac{1}{A^2}\frac{\ddot{f}_2(T)}{f_2(T)}\left(\frac{dT}{dx}\right)^2+\frac{1}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{d^2T}{dx^2}+\frac{\alpha}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dx}=0$$

And above equations (54) reduces to

$$\frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}+\frac{\dot{C}}{C}\left[\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right]+\left[\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right]\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2T)}{f_2(T)}\frac{dT}{dt}\right]=0 \quad (56a)$$

$$\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}+\frac{\dot{C}}{C}\left[\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right]+\left[\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right]\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2T)}{f_2(T)}\frac{dT}{dt}\right]=0 \quad (56b)$$

Integrating (56), we get

$$\frac{C}{B}=m_6\exp\left\{m_5\int\frac{1}{BC^2\dot{f}_1(R)f_2(T)}dt\right\} \quad (57a)$$

$$\frac{B}{C}=m_8\exp\left\{m_7\int\frac{1}{BC^2\dot{f}_1(R)f_2(T)}dt\right\} \quad (57b)$$

Where, $m's$ be constants of integration with the condition that $m_6m_8=1$ and $m_5+m_7=0$

From (57) we can express the values of metric functions explicitly as

$$C=(BC^2)^{\frac{1}{3}}m_9exp\left\{m_{10}\int\frac{1}{BC^2\dot{f}_1(R)f_2(T)}dt\right\} \quad (58a)$$

$$B=(BC^2)^{\frac{1}{3}}m_{11}exp\left\{m_{12}\int\frac{1}{BC^2\dot{f}_1(R)f_2(T)}dt\right\} \quad (58b)$$

From (55) and (58a)

$$A=(BC^2)^{\frac{1}{3}}m_{13}\exp\left\{m_{10}\int\frac{1}{BC^2\dot{f}_1(R)f_2(T)}dt\right\} \quad (58c)$$

We can rewrite (58) as

$$A=(ABC)^{\frac{1}{3}}m_{14}\exp\left\{m_{15}\int\frac{1}{ABC\dot{f}_1(R)f_2(T)}dt\right\} \quad (59a)$$

$$B=(ABC)^{\frac{1}{3}}m_{16}\exp\left\{m_{17}\int\frac{1}{ABC\dot{f}_1(R)f_2(T)}dt\right\} \quad (59b)$$

$$C=(ABC)^{\frac{1}{3}}m_{18}\exp\left\{m_{19}\int\frac{1}{ABC\dot{f}_1(R)f_2(T)}dt\right\} \quad (59c)$$

Where, $m's$ be constants of integration.

Adjusting the constants in (36) and (59), the line element (15) assumes an isotropic form and hence, we generalize the result in the form of following theorem,

Theorem 1: In f(R, T) gravity theory, the Bianchi type V space-time filled with scalar field coupled with electromagnetic field, admits isotropy for the functional form $f(R,T)=f_1(R)+\lambda f_2(T)$ and $f(R,T)=f_1(R)f_2(T)$

With the help of (47) the equations (23) can be written as

$$\left(\frac{\dot{h}_7}{h_7}\right)^{\cdot}+\frac{\dot{h}_7^{\,2}}{h_7^{\,2}}+\frac{\dot{h}_7}{h_7}\left[\frac{\dot{C}}{C}+\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right]=\dot{\psi}\dot{\varphi}^2 \quad (60a)$$

$$\left(\frac{\dot{h}_7}{h_7}\right)^{\cdot}+\frac{\dot{h}_7^{\,2}}{h_7^{\,2}}+\frac{\dot{h}_7}{h_7}\left[\frac{\dot{A}}{A}+\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right]=\dot{\psi}\dot{\varphi}^2 \quad (60b)$$

$$\left(\frac{\dot{h}_7}{h_7}\right)^{\cdot}+\frac{\dot{h}_7^{\,2}}{h_7^{\,2}}+\frac{\dot{h}_7}{h_7}\left[\frac{\dot{B}}{B}+\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right]=\dot{\psi}\dot{\varphi}^2 \quad (60c)$$

These equations further imply that

$$\frac{\dot{C}}{C}+\frac{\dot{A}}{A}-\frac{\dot{B}}{B}=\frac{\dot{A}}{A}+\frac{\dot{B}}{B}-\frac{\dot{C}}{C}=\frac{\dot{B}}{B}+\frac{\dot{C}}{C}-\frac{\dot{A}}{A}$$

or $$\frac{\dot{A}}{A}=\frac{\dot{B}}{B}=\frac{\dot{C}}{C} \quad (61)$$

Inserting (61) in (60) yield

$$\left(\frac{\dot{h}_7}{h_7}\right)^{\cdot} + \frac{\dot{h}_7{}^2}{h_7{}^2} + \frac{\dot{h}_7}{h_7}\left[\frac{\dot{A}}{A}\right] = \dot{\psi}\dot{\varphi}^2 \qquad (62)$$

From (48) $\dot{\psi}\dot{\varphi}^2 = \frac{\dot{h}_7{}^2}{h_7{}^2} + \frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2$ (63)

Inserting (63) in (62), we obtain

$$\left(\frac{\dot{h}_7}{h_7}\right)^{\cdot} + \frac{\dot{h}_7}{h_7}\left[\frac{\dot{A}}{A}\right] = \frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2 \qquad (64)$$

By confining the linearity of the function ψ i.e. $\ddot{\psi} = 0$ or $\psi = m_{20}I + m_{21}$ the equation (64) has solution

$$h_7 = m_{23}\exp\left\{m_{22}\int \frac{1}{A}dt\right\} \qquad (65)$$

Hence, with the help of (65) the equations (49) convert into

$$V_1 = m_{24}\exp\left\{m_{22}\int \frac{1}{A}dt\right\} \qquad (66a)$$

$$V_2 = m_{25}\exp\left\{m_{22}\int \frac{1}{A}dt\right\} \qquad (66b)$$

$$V_3 = m_{26}\exp\left\{m_{22}\int \frac{1}{A}dt\right\} \qquad (66c)$$

Where, $m's$ be constant of integration.

Adjusting the constants in (49) and (66) the vector potential assume the following form

$V_i = [V_1(t), V_1(t), V_1(t), 0]$

Hence, we generalize the result in the form of following theorem.

Theorem 2: In the f(R, T) theorem of gravity the Bianchi type V space-time filled with scalar field coupled with electromagnetic field admits the vector potential $V_i = [V_1(t), V_1(t), V_1(t), 0]$ for the functional form $f(R,T) = f_1(R) + \lambda f_2(T)$ and $f(R,T) = f_1(R)f_2(T)$

Sub Case $f(R,T) = f(R)$

Here, we follow the notations $f(R,T) = f(R)$, $f_R(R,T) = \frac{\partial f(R,T)}{\partial R} = \dot{f}(R)$, $f_T(R,T) = \frac{\partial f(R,T)}{\partial T} = 0$

In this case equation (5) reduces to

$$G_j^i = \frac{1}{\dot{f}(R)}\left[g^{im}\nabla_m\nabla_j\dot{f}(R)\right] - \frac{1}{6\dot{f}(R)}\left[\dot{f}(R)R + f(R)\right]g_j^i + \frac{\chi}{\dot{f}(R)}\left[T_j^i - \frac{1}{3}Tg_j^i\right] \qquad (67)$$

The computation for this case easily follows from those of the earlier case $f(R,T) = f_1(R) + \lambda f_2(T)$ (section 4) by mere substitution of $f_1(R) = f(R)$, $\lambda = 0$ or $f_2(t) = 0$

We get the result as follows

$$A = (ABC)^{\frac{1}{3}}k_{27}\exp\left\{k_{28}\int \frac{1}{ABC\dot{f}(R)}dt\right\} \qquad (68a)$$

$$B = (ABC)^{\frac{1}{3}}k_{29}\exp\left\{k_{30}\int \frac{1}{ABC\dot{f}(R)}dt\right\} \qquad (68b)$$

$$C = (ABC)^{\frac{1}{3}}k_{31}\exp\left\{k_{32}\int \frac{1}{ABC\dot{f}(R)}dt\right\} \qquad (68c)$$

Where, $k's$ are constants of integration.

$$V_1 = k_{35}\exp\left\{k_{33}\int \frac{1}{A}dt\right\} \qquad (69a)$$

$$V_2 = k_{36}\exp\left\{k_{33}\int \frac{1}{A}dt\right\} \qquad (69b)$$

$$V_3 = k_{37}\exp\left\{k_{33}\int \frac{1}{A}dt\right\} \qquad (69c)$$

Where, $k's$ are constants of integration.

From section IV, V, and VI we observe that the result remain intact for $f(R,T) = f_1(R) + \lambda f_2(T)$ $f(R,T) = f_1(R)f_2(T)$, $f(R,T) = f(R)$, differ in constants of integration only. Hence, equations (68) and (69), admits the theorems 1 and 2.

Sub Case $f(R,T) = R + \lambda T$

Here, we follow the notations

$$f(R,T) = R + \lambda T, \quad f_R(R,T) = \frac{\partial f(R,T)}{\partial R} = 1, \quad f_T(R,T) = \frac{\partial f(R,T)}{\partial T} = \lambda$$

In this case the field equation (5) reduces to

$$G_j^i = \chi T_j^i - \lambda\left[T_j^i + \theta_j^i\right] + \frac{\lambda}{2} T g_j^i \qquad (70)$$

The consideration of this sub case follows from (section 4) $f(R,T) = f_1(R) + \lambda f_2(T)$ by taking $f_1(R) = R$ and $f_2(T) = T$

We get the result as follows

$$A = (ABC)^{\frac{1}{3}} l_{11} \exp\left\{l_{12} \int \frac{1}{ABC} dt\right\} \qquad (71a)$$

$$B = (ABC)^{\frac{1}{3}} l_{13} \exp\left\{l_{14} \int \frac{1}{ABC} dt\right\} \qquad (71b)$$

$$C = (ABC)^{\frac{1}{3}} l_{15} \exp\left\{l_{16} \int \frac{1}{ABC} dt\right\} \qquad (71c)$$

Where, $l's$ are constant of integration

$$V_1 = l_{21} \exp\left\{l_{20} \int \frac{1}{A} dt\right\} \qquad (72a)$$

$$V_2 = l_{22} \exp\left\{l_{20} \int \frac{1}{A} dt\right\} \qquad (72b)$$

$$V_3 = l_{23} \exp\left\{l_{20} \int \frac{1}{A} dt\right\} \qquad (72c)$$

Where, $l's$ be constant of integration.

From section IV, V and VII we observe that the result remain intact for $(R,T) = f_1(R) + \lambda f_2(T)$, $f(R,T) = f_1(R) f_2(T)$ and $f(R,T) = R + \lambda T$, only differ in the constant of integration. Hence, equations (71) and (72) admits the theorem 1 and 2.

Sub Case $f(R,T) = f(R) = R$

In this *case* $f_R(R,T) = f_R(R) = \frac{\partial f(R)}{\partial R} = 1$ $f_T(R,T) = \frac{\partial f(R)}{\partial T} = 0$

In this case the field equation (5) reduces to

$$G_j^i = \chi T_j^i \qquad (73)$$

The computation for this case easily follows from those of the earlier case (section VII) $f(R,T) = R + \lambda T$ by mere substitution of $\lambda = 0$

We get the result as follows

$$A = (ABC)^{\frac{1}{3}} l_{33} \exp\left\{l_{34} \int \frac{1}{ABC} dt\right\} \qquad (74a)$$

$$B = (ABC)^{\frac{1}{3}} l_{35} \exp\left\{l_{36} \int \frac{1}{ABC} dt\right\} \qquad (74b)$$

$$C = (ABC)^{\frac{1}{3}} l_{37} \exp\left\{l_{38} \int \frac{1}{ABC} dt\right\} \qquad (74c)$$

Where, $l's$ be constant of integration.

$$V_1 = l_{41} \exp\left\{l_{39} \int \frac{1}{A} dt\right\} \qquad (75a)$$

$$V_2 = l_{42} \exp\left\{l_{39} \int \frac{1}{A} dt\right\} \qquad (75b)$$

$$V_3 = l_{43} \exp\left\{l_{39} \int \frac{1}{A} dt\right\} \qquad (75c)$$

Where, $l's$ be constants of integration. From section IV, V and VIII, we observe that the result remain intact for $(R,T) = f_1(R) + \lambda f_2(T)$, $f(R,T) = f_1(R) f_2(T)$ and $f(R,T) = R$, only differ in the constant of integration. Hence, equations (74) and (75) admits the theorem 1 and 2.

Sub Case $f(R,T) = RT$

In this case we follow the notations $f_R(R,T) = \frac{\partial f(R,T)}{\partial R} = T$, $f_T(R,T) = \frac{\partial f(R,T)}{\partial T} = R$ then, equation (8) reduces to

$$G_j^i = \frac{1}{T}\left[g^{im} \nabla_m \nabla_j T\right] - \frac{1}{3} R g_j^i + \frac{\chi}{T}\left[T_j^i - \frac{1}{3} T g_j^i\right] + \frac{1}{3}\frac{R}{T}[T + \theta] g_j^i - \frac{R}{T}\left[T_j^i + \theta_j^i\right] \qquad (76)$$

The computation for this case easily follows from those of the earlier case (section V) by mere substitution of $f_1(R) = R$ and $f_2(T) = T$

We get the result as follows

$$A = (ABC)^{\frac{1}{3}} n_{15} \exp\left\{n_{16} \int \frac{1}{ABCT} dt\right\} \quad (77a)$$

$$B = (ABC)^{\frac{1}{3}} n_{17} exp\left\{n_{18} \int \frac{1}{ABCT} dt\right\} \quad (77b)$$

$$C = (ABC)^{\frac{1}{3}} n_{19} exp\left\{n_{20} \int \frac{1}{ABCT} dt\right\} \quad (77c)$$

Where, $n's$ are constants of integration.

$$V_1 = n_{25} \exp\left\{n_{23} \int \frac{1}{A} dt\right\} \quad (78a)$$

$$V_2 = n_{26} \exp\left\{n_{23} \int \frac{1}{A} dt\right\} \quad (78b)$$

$$V_3 = n_{27} \exp\left\{n_{23} \int \frac{1}{A} dt\right\} \quad (78c)$$

Where, $n's$ be constants of integration. From section IV, V and VI, we observe that the result remain intact for $f(R,T) = f_1(R) + \lambda f_2(T)$, $f(R,T) = f_1(R) f_2(T)$ and $f(R,T) = RT$ only differ in constants of integration. Hence, equations (77) and (78) admit the theorem 1 and 2.

Conclusion

1) In the present paper we have considered sub cases of $f(R,T)$ theory of gravity models

(i) $f(R,T) = f_1(R) + \lambda f_2(T)$
(ii) $f(R,T) = f(R)$
(iii) $f(R,T) = R + \lambda T$
(iv) $f(R,T) = R$
(v) $f(R,T) = f_1(R) f_2(T)$
(vi) $f(R,T) = RT$

In Bianchi type V metric. We have derived the gravitational field equations corresponding to the general and particular cases of $f(R,T)$ theory of gravity.

2) It is observed that, even though the cases of $f(R,T)$ theory are distinct, the convergent, non-singular, isotropic solutions can be evolved in each case along with the components vector potential.

3) From finding of the $f(R,T)$ and $f(R)$ theory, general and particular cases, in this paper we believe firmly that the results of $f(R,T)$ and $f(R)$ depends on only R and not on T

4) From different cases of $f(R,T)$ we observe that the results remain intact only differ in constants of integration.

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