Indian Journal of Plant Sciences ISSN: 2319–3824(Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jps.htm
2017 Vol.6 (2) April-June, pp. 37-51/Mahanta and Borah
Research Article

COMPARATIVE STUDY OF CHAPMAN- RICHARDS GROWTH MODEL AND IT'S LIMITING CASES FOR GROWTH OF BABUL (*ACACIA NILOTICA*) TREES

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ABSTRACT
The main objective of this study is to find the best fit growth model along with the best method of estimation for the babul growth in India. This study presents a comparative study among Chapmen Richards's growth models and its various limiting cases. A number of established methods of estimation have been considered to estimate the parameters of Logistic, Gompertz, Monomolecular growth model. Few estimation techniques are also introduced for Chapmen Richards, Von Bertalanffy and Negative Exponential growth model. The methods of estimation use in this study demands less computations and can also deals with data without any regular spacing time. This study will help to find out the best fit methods of estimations along with the growth model from forestry viewpoint. Two sets of well-established babul (*Acacia Nilotica*) tree growth data of India have been used for testing the validity of the models. The best fit model has been selected based on a selection criterion. In case of top height growth data of babul tree, the Monomolecular growth model is found to be more suitable as the value of $R^2_{prediction}$ and R^2 (99.99 and 99.99 respectively) are better than the remaining growth models. Monomolecular growth model also provide the better results with the $R^2_{prediction}$ and R^2 values 99.97 and 99.99 respectively for the maximum diameter growth of babul in India. According to the results, it can be concluded that, the three parameter Monomolecular growth model is more reasonable over the remaining growth models to describe the growth of Babul in India.

Keyword: *Growth Model, Chapmen Richards, Von Bertalanffy, Logistic, Gompertz, Monomolecular, Negative Exponential, Acacia Nilotica*

INTRODUCTION
Babul (*Acacia Nilotica*) is a multipurpose tree native to Africa, the Middle East and the Indian subcontinent. Its timber is valued by rural folks, its leaves and pod are used as food and gum has a number of uses. Though it is not as long-lasting as teak wood, furniture made from babul wood can still last for many years and hence, serves as a cheaper alternative to teak wood. Babul wood furniture can also be used in the open air as the wood has good resistance to water and climatic changes. Other than furniture, the wood obtained from the tree is mainly used for making pulpwood and also for medicinal purposes. The modeling methodology is a powerful tool for the study of growth. It provides smooth curves of age and growth, even from irregularly spaced measurements. The comparison between families of curves can be done using parameter estimates. In this study Chapmen Richards's growth models and its families are considered for a comparative study of growth of babul tree in India. Its family mainly consists of five another growth models, namely Von Bertalanffy, Logistic, Gompertz, Monomolecular and Negative Exponential growth model. The Chapman Richards growth model was first used for forest growth modeling studies by Turnbull in 1963 and then by Pienaar in 1965 (Yuancai *et al.*, 1997). The limiting cases of the Chapman Richards growth model also play a key role in forestry study. Many authors used these models to study the forest system in various time (Borah and Mahanta, 2013; Fekedulegn *et al.*, 1999; Zhao-gang and Feng-ri, 2003; Mayers *et al.*, 2012; Oliver, 1964). These models can be also used in other fields of science. These models were used to study the oldest-old mortality rates by Saikia and Borah (2013, 2014), Kucuk and Eydura (2009) used these to study Akkaraman and German

Blackheaded Mutton X Akkaraman B_1crosbreed lambs. To study the weight of Norduz female lambs, Kum *et al.*, (2010) used these models.
The properties and the derivations of the growth models play a crucial rule for estimating the parameters. Proper understanding of the mathematics of these models avoids problems encountered in the method of parameter estimations of the models. This paper provides the most fundamental properties along with some useful definition of the parameters. This paper applies some new technique to estimate the parameters of negative exponential, Von Bertalanffy and Chapmen Richards's growth models. The main advantages of the new techniques applied in this paper are less computation and can be used for any growth data. The parameters of Monomolecular, Logistic and Gompertz growth models are estimated using the technique used by Borah and Mahanta (2013).

MATERIALS AND METHODS

The growth models considered for this study were Chapmen Richards, Von Bertalanffy, Logistic, Gompertz, Monomolecular and Negative Exponential growth model. The integral forms of these models were shown in the Table 1. For these models consider, y was the dependent growth variable, t was the independent variable, A, B, K, d, β, b_1 and m were parameters to be estimated and $\exp(e)$ was the base of the natural logarithms. The properties of the parameters of this model were discussed in this paper. The maximum diameter data and top height growth of *babul (Acacia Nilotica) tree* were based on the analysis of sample plot data of Uttar Pradesh, Maharashtra and Madhya Pradesh (Indian Council of Forestry Research and Education, No Date). The data sets were presented in Table 2 and Table 3.
The growth models can be written in the form as

$$y_i = f(t_i, \mathrm{B}) + \varepsilon_i, \qquad (1)$$

$i = 1,2, \cdots, n,$ where, $B$was the vector of parameters b_j $(b_1, b_2, \cdots, b_s)$ to be estimated. Where, s was the number of parameter, n was the number of observations and ε_i's were random errors in the models had mean zero and constant variance σ^2. The parameters of the growth models were defined as: A was the asymptote; $K$was the parameter governing the rate at which the regressand approaches its potential maximum; m was the allometric constant; $d$was the instant rate of growth in the inflection point and B, β and b_1 were biological constants. The following methods of estimation had been used to fit the growth models.

Table 1: List of Growth Models with their Integral Forms

S/N	Common Growth Models	Integral Form of the Models ($y(t)$)	Derivation from Chapman Richards Model	Source
1	Chapmen Richards	$A\{1 - Be^{-Kt}\}^d$	$d = \dfrac{1}{1-m}$	Ozel *et al.*, (2010)
2	Von Bertalanffy	$\{A^{1-m} - b_1 e^{-Kt}\}^{\frac{1}{1-m}}$	$b_1 = A^{1-m}B$	Colbert *et al.*, (2003)
3	Logistic	$\dfrac{A}{1+\beta e^{-Kt}}$	$m = 2$ and $\beta = -B$	Borah and Mahanta (2013)
4	Gompertz	$Ae^{-Be^{-Kt}}$	$m \to 1$	Borah and Mahanta (2013)
5	Monomolecular	$A(1 - Be^{-Kt})$	$m = 0$	Borah and Mahanta (2013)
6	Negative Exponential	$A(1 - e^{-Kt})$	$B = 1$ and $m = 0$	Philip (1994)

Table 2: Top Height Growth Data of Babul Tree in India

Age (year)	5	10	15	20	25
Top height(m)	8.14	12.19	14.93	16.70	17.98

Table 3: Maximum Diameter Growth Data of Babul Tree in India

Age (years)	5	10	15	20	25
Maximum diameter (cm)	12.19	20.83	26.92	31.49	34.29

Method of Estimation

The parameters of Monomolecular, Logistic and Gompertz growth had been estimated using the technique used by Borah and Mahanta (2013). The new methods for the estimation of the parameters of Von Bertalanffy and Chapmen Richards growth models were described as follows.

Methods to Estimate the Parameters of Chapmen Richards's Growth Model

Method A: In this method, first assume that the parameter d was known from its definition. Then, let n be the total number of observation and let t_a,t_b and t_c were any three observations from the set of data. Then, for $i = a, b, c$; the Chapmen Richards growth model could be written as

$$\ln y_i = \ln A + d \ln(1 - Be^{-Kt_i}). \tag{2}$$

Now,

$$\ln y_a - \ln y_b = d \ln\left\{\frac{1 - Be^{-Kt_a}}{1 - Be^{-Kt_b}}\right\}, \tag{3}$$

and

$$\ln y_b - \ln y_c = d \ln\left\{\frac{1 - Be^{-Kt_b}}{1 - Be^{-Kt_c}}\right\}. \tag{4}$$

From the equation (3) and (4),

$$(A_1A_2 - A_2)x^{t_c} + (1 - A_1A_2)x^{t_b} + (A_2 - 1)x^{t_a} = 0, \tag{5}$$

Where $A_1 = \exp\frac{\ln y_a - \ln y_b}{d}$ and $A_2 = \exp\frac{\ln y_b - \ln y_c}{d}$

The equation (5) could be solved using any iteration method, and then the parameter K could be estimated as,

$$\hat{K} = \ln\frac{1}{x}.$$

After estimating the parameter K; the parameters B, A and d could be estimated using the equations (2), (3) and (4). And the parameters were given by

$$\hat{B} = \frac{1 - \exp\frac{\ln y_b - \ln y_c}{d}}{\exp(-Kt_b) - \exp\frac{\ln y_b - \ln y_c}{d}\exp(-Kt_c)},$$

$$\hat{A} = \exp\{\ln y_c - r\ln(1 - Be^{-Kt_c})\},$$

$$\hat{d} = \frac{\ln\ y_a - \ln A}{\ln(1 - Be^{-Kt_a})}.$$

For some equidistant data set, consider $\boldsymbol{r = \left[\frac{n}{3}\right]}$, $\boldsymbol{t_a = r, t_b = 2r}$ and $\boldsymbol{t_c = 3r}$. In this case the equation obtained by solving equations (3) and (4) would be in quadratic form with $\boldsymbol{x = e^{-rK}}$.

Method B: For this method, assume that the parameter B and K were known. Let n be the total number of observation and $r = \left[\frac{n}{2}\right]$, $S_1 = \sum_{i=1}^{r} \ln y_i$ and $S_2 = \sum_{i=r+1}^{2r} \ln y_i$, then the estimated parameters were given by

$$\hat{d} = \frac{S_2 - S_1}{\left\{\ln\left(\prod_{i=r+1}^{2r}(1 - Be^{-Kt_i})\right) - \ln\left(\prod_{i=1}^{r}(1 - Be^{-Kt_i})\right)\right\}},$$

$$\hat{A} = \exp\left\{\frac{S_1 - d\ln(\prod_{i=1}^{r}(1 - Be^{-Kt_i}))}{r}\right\}.$$

Research Article

After estimating the parameters A and d; the parameter K and B could be estimated as

$$\widehat{K} = \frac{1}{\sum_{i=r+1}^{2r} t_i - \sum_{i=1}^{r} t_i} \ln \left[\frac{\prod_{i=1}^{r} \left(1 - \left(\frac{y_i}{A}\right)^{\frac{1}{d}}\right)}{\prod_{i=r+1}^{2r} \left(1 - \left(\frac{y_i}{A}\right)^{\frac{1}{d}}\right)} \right],$$

$$\widehat{B} = \exp\left\{\frac{K}{n}\sum_{i=1}^{n} t_i + \frac{1}{n}\ln\left\{\prod_{i=1}^{n}\left(1 - \left(\frac{y_i}{A}\right)^{\frac{1}{d}}\right)\right\}\right\}.$$

Methods to Estimate the Parameters of Von Bertalanffy Growth Model

The von Bertalanffy growth model could also be written as

$$y = (A^{1-m} - b_1 e^{-Kt})^{\frac{1}{1-m}},$$

$$y = A\left(1 - Be^{-kt}\right)^d. \tag{6}$$

Where, $B = \frac{b_1}{A^{1-m}}$ and $d = \frac{1}{1-m}$.

The equation (6) is in the form of Chapmen Richards's growth model and its parameters could be estimated using the same methodology as Chapmen Richards's growth model. After estimating the parameters the required parameters could be estimated using $b_1 = BA^{1-m}$ and $m = \frac{d-1}{d}$.

Methods to Estimate the Parameters of Negative Exponential Growth Model

Method A: Let n be the total number of observation. Let t_1 be the first observation and t_2 be the n^{th} observation. Then, the Negative Exponential growth equation could be written as

$$y_1 = A(1 - e^{-K}), \tag{7}$$

$$y_2 = A(1 - e^{-Kn}), \tag{8}$$

Now by solving equation (7) and (8)

$$y_1 x^n - y_2 x + (y_2 - y_1) = 0. \tag{9}$$

Where, $x = e^{-K}$. The equation (9) could be solved using any iteration method. After finding the value of x, the parameter could be estimated using $K = -\ln x$. Then, the parameter A could be estimated using the equation (7) or (8).

Now since the iteration method need an initial value. For initial value approximation the following procedure was used:

For the first and second data of the data set, the equation (9) could be written as

$$y_1 x^2 - y_2 x + (y_2 - y_1) = 0, \tag{10}$$

which was a quadratic equation. The non-negative value(s) of x could be used as starting value.

Method B: In this method assume that the parameter A was known from the previous method. Then, rewriting the model in terms of K and then considering the sum of all observations,

$$K = \frac{1}{n}\ln\left[\prod_{i=1}^{n} \frac{1}{\left(1 - \frac{y_i}{A}\right)^{\frac{1}{t_i}}}\right]; i = 1, 2, \cdots, n \tag{11}$$

After estimating K, again rewriting the model in terms of A and adding for the entire observations, the parameter A could be estimated, that is

$$A = \frac{1}{n}\sum_{i=1}^{n} \frac{y_i}{1 - e^{-Kt_i}};\ \ i = 1, 2, \cdots, n \tag{12}$$

Properties of the Growth Models

There is a clear relationship between the properties of different mathematical models and the estimation of their respective parameters. If the properties of nonlinear mathematical models were to be known then it may pretty helpful to estimate the parameters to be estimated. Indeed, even a few cases, because of absence of knowledge of these properties, it might appear to face different problems to use in different natural growth (Mahanta and Borah, 2014). Some basic properties of the mentioned model were provided in table 4.

Selection Criteria of Best Fit Model

After fitting the growth models using different methods of estimation, the best fit model was selected based on the following selection criteria. The selection criteria consist of six distinct steps.

Step I: Logical and Biological Consistency: in this step, check the logical consistent and biologically realistic of the estimated parameters. The growth models with non-consistent and non-natural consistency and poor statistical properties were excluded.

Step II: Chi-Square Goodness-of-Fit Test (χ^2)*:* This test enables to see how well does the growth model fit to the observed data. The Chi-Square was defined as

$$\chi^2 = \sum_{i=1}^{n} \frac{(y_i - \hat{y}_i)^2}{\hat{y}_i},$$

Where, y_i was the observed value and $\hat{y}_i$ was the predicted value for $i = 1, 2, \cdots, n$. If the calculated value of χ^2 was greater than the tabulated value of χ^2 with $n - 1 - p$ degrees of freedom (where p was the number of parameters of the growth model and n was the number of observations) then the null hypothesis was rejected otherwise accepted. In this literature, only those results were considered which have 95% level of significance with their respective degree of freedom.

Step III: The Root Mean Square Error (RMSE): The RMSE measures to aggregate the residuals into one measure of predictive power. The RMSE of a model prediction with reference to the calculable variable was defined as:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n}},$$

Where, y_i was the observed values and $\hat{y}_i$ was the predicted values for $i = 1, 2, \cdots, n$. By comparing the RMSE the ten best results were selected.

Step IV: Coefficient of Determination (R^2) *and Adjusted Coefficient of Determination* (R_a^2)*:* The R^2 value indicates how well data point fits a growth model. Generally, the value of (R^2) lies between 0 and $1 (0 \leq R^2 \leq 1)$. But it is impossible for R^2 to actually attain 1, if pure error exist. In practice, sometime negative value of R^2 may occur. Theoretically the value 1 indicates a perfect fit, 0 reveals that the model is not a better than the simple average and negative value indicate a poor model (Soares *et al.*, 1995). If the value of R^2 is above 0.9, it is accepted as efficient (Ozel *et al.*, 2010). The mathematical formulation of the coefficient of determination was,

$$R^2 = 1 - \frac{\sum(y_i - \hat{y}_i)^2}{\sum(y_i - \bar{y})^2},$$

Where, $\bar{y}$ is the mean of the response variables. The R_a^2 value was an endeavor to redress the propensity for over fitting of R^2 by adjusting both the numerator and the exterminator by their respective degree of freedom and defined as

$$R_a^2 = 1 - (1 - R^2)\left(\frac{n-1}{n-p}\right).$$

The R_a^2 were used to compare growth models not only to a specific set of data but also to two or more entirely different sets of data. The equation with the least standard error of the estimate will most likely also have the maximum R_a^2. In this manuscript, only those results were considered which had R_a^2 value not less than 0.99.

Step V: Confidence Interval: In this step, the confidence intervals of the estimated parameters were found. Let B is the vector of the parameters (say the parameters are $\beta_1, \beta_2, \cdots, \beta_p$) of the growth models. Confidence limits for the true value of the parameters B were evaluated on the basis of the linearized approximation, evaluated at the predicted value of the parameters$(\hat{B})$. The $100(1 - \alpha)\%$ confidence interval for the parameters B was

$$\hat{B}_i \pm t_{\frac{\alpha}{2},n-p}\, se\left(\hat{B}_i\right),$$

Where $t_{\frac{\alpha}{2},n-p}$ was the $t - value$ at $n - p$ degrees of freedom and $se\left(\hat{B}_i\right) = \left\{appropriate\ diagonal\ element\ of\left(\hat{Z}'\hat{Z}\right)^{-1} S^2\right\}^{\frac{1}{2}}$. Here $S^2 = S(\hat{B})/(n-p)$; $S(\hat{B}) = \sum_{i=1}^{n}\left\{y_i - f(t_i,\hat{B})\right\}^2$ and

$$\hat{Z} = \begin{bmatrix} \frac{\partial f(t_1,B)}{\partial \beta_1} & \frac{\partial f(t_1,B)}{\partial \beta_2} & \cdots & \frac{\partial f(t_1,B)}{\partial \beta_p} \\ \frac{\partial f(t_2,B)}{\partial \beta_1} & \frac{\partial f(t_2,B)}{\partial \beta_2} & \cdots & \frac{\partial f(t_2,B)}{\partial \beta_p} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f(t_n,B)}{\partial \beta_1} & \frac{\partial f(t_n,B)}{\partial \beta_2} & \cdots & \frac{\partial f(t_n,B)}{\partial \beta_p} \end{bmatrix}.$$

The final estimate of the parameters with$\sim$95% confidence band excluding zero, indicating that there were only non-zero values of the parameters and then they were always significant. In this step, those results with negative confidence interval had been eliminated.

Step VI: Approximate R^2 for Prediction: Finally, calculate the approximate R^2 for prediction and it was given by

$$R^2_{prediction} = 1 - \frac{PRESS}{\sum(y_i - \bar{y})^2},$$

Where $PRESS = \sum_{i=1}^{n}\left(\frac{e_i}{1-h_{ii}}\right)^2$ was known as the PRESS statistic. Here, $e_i = y_i - \hat{y}_i$ and h_{ii} were the diagonal elements of hat matrix $H = T(T'T)^{-1}T'$ and T was a $n \times 1$ matrix of the independent variables. This statistic gives some indication of the predictive capability of the model. If the value of $R^2_{prediction}$ was r and the value of $R^2$was m, then one could expect from the model to explain about r% of the variability in predicting new observations, as compared to the approximately m% of the variability in the original data explained by the fitting (Mayers *et al.*, 2012). Base on this statistics, the best fit model for different growth of babul tree in India was selected.

RESULTS AND DISCUSSION

Chapmen Richard's growth models and its various limiting cases have been fitted to top height and maximum diameter growth data of babul trees compiled from Uttar Pradesh, Maharashtra and Madhya Pradesh of India.

The parameters of these models are estimated using a total of twenty four methods of estimation. The estimation of parameters for the growth models along with the summary of statistical analysis to top height growth data of babul tree are presented in Table 5. Based on six model selection criteria as discussed above we summarized the results as bellow.

Step I: The Logistic model estimated by method B and D are rejected due to non-logical estimation of the parameters. All the methods have estimated the asymptotes smaller than the dominant height of babul tree (17.98m). The estimated parameters of the rest of the models are logically consistent and biologically significant.

Step II: The chi square test is not applicable for Chapmen Richards and Von Bertalanffy growth model as this study used a data set with five observation and the both models has four parameters each. In that case, the degree of freedom becomes zero and there is no any chi square value for zero degree of freedom. Based on step II, Gompertz growth model (method B, D and E) and Logistic model (method A, C, E and F) are rejected due to having less than 95% level of significance.

Step III: Considering the relative value of RMSE, the ten best results have been selected in this step. Comparing the value of RMSE, Monomolecular growth models with all its methods of estimation, Gompertz growth model with method F, Chapmen Richards model for method B and Von Bertalanffy growth model (Method A and B) are promoted for the next level.

Step IV: In the fourth step, no growth result has been eliminated as all surviving results have R_a^2 value 0.99.

Step V: All surviving results along with the 95% confidence level are demonstrated in Table 6. It is observed that the Von Bertalanffy growth model (method A and B) is removed as some of the parameters of this model are not significantly different from zero at 95% confidence level.

Step VI: The sixth and final selection criterion is based on $R^2$and $R_{prediction}^2$, as this statistic gives some indication of the predictive capability of the growth models. From the final step, we select the best growth model.

In case of top height growth data of babul tree, the Monomolecular growth model (methods A, C, D, E and F) is found to be more suitable as the value of $R_{prediction}^2$ and R^2 (99.99 and 99.99 respectively) are better than the remaining surviving growth models. The observed and the estimated values are shown in figure 1. The eliminated results in each step are highlighted accordingly in the Table 5.The estimation of parameters for the growth models and the summary of statistical analysis is to maximum diameter growth data of babul tree are presented in Table 7. In this case, logistic growth model (method B and D) and Gompertz growth model (method B) are rejected due to non-logical estimation of the parameters. In all the cases, some of their parameters estimate of asymptotic parameters smaller than the dominant diameter of babul tree (34.29cm). The eliminated results in each step are also highlighted accordingly in the Table 7. For maximum diameter growth data of babul tree, the chi square test is also not applicable for Chapmen Richards and Von Bertalanffy growth model as both have zero degree of freedom. Gompertz growth model (method A, C, D, E and F) and Logistic model (method A, C, E and F) are rejected due to having less than 95% level of significance. In the third step, comparing the value of RMSE, Monomolecular growth models with all its methods of estimation, negative exponential growth model (method A and B), Chapmen Richards model (method A) and Von Bertalanffy growth model (Method B) are promoted for the next level. The Von Bertalanffy model (method B) is eliminated in the step V, as some of the parameters of this model are not significantly different from zero at 95% confidence level (Table 6). And finally, we choose the best fit model and find that Monomolecular growth model with method C, D, E and F give the similar results with the $R_{prediction}^2$ and R^2 values 99.97 and 99.99 respectively. The two results are plotted in order to illustrate their differences (Figure 2). All the results produced a very similar result for maximum diameter growth data of babul tree in India. There might be more than one model that to be regarded as 'useful'. It means that the data are inadequate and ambivalent concerning some impact or parameterization or structure. It is reasonable that several models would serve almost similarly well in approximating a set of data. There is often considerable uncertainty in the choice of a specific model as the "best" approximating model (Burnham and Anderson, 1998). However, from the results presented in this paper, it is clearly observed that for top height growth data of babul tree, Monomolecular growth model along with methods A, C, D, E and F (value of $R_{prediction}^2$ and R^2 are 99.99 and 99.99 respectively) is found to be more suitable than the remaining growth models whereas Monomolecular growth model with method C, D, E and F ($R_{prediction}^2$ and R^2 values are 99.97 and 99.99 respectively)provides a better fit for maximum growth data. In the study by Tewari *et al.*, (2007), calibrate three growth models to the height growth of *Acacia Nilotica* in Gujarat state in India. A similar study was also done by Abakar and Ahmed (2014) for the growth of the species in Riverine Forests - Blue Nile. In both work, the parameters were estimated using two different methods of estimation, which involve minimizing the objective function using nonlinear optimization technique. The newly introduced methods of estimation present in this paper demands less computation and can use any growth data. In contrast, modern statistical methods required some depth knowledge of mathematics. This paper shows some simple method for dealing with data without a regular spacing time. This will provide some simple tools for researchers with limited experience in the application of more complex models. This study will help the researchers in the area of forestry and mathematical modeling.

Indian Journal of Plant Sciences ISSN: 2319–3824(Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jps.htm
2017 Vol.6 (2) April-June, pp. 37-51/Mahanta and Borah
Research Article

Table 4: Summary of Some Basic Properties of the Growth Models

	Chapman Richards	**Von Bertalanffy**	**Logistic**	**Gompertz**	**Monomolecular**	**Negative Exponential**
Integral form of the growth function	$y(t) = A\{1 - Be^{-Kt}\}^d$	$y(t) = \{A^{1-m} - b_1 e^{-Kt}\}^{\frac{1}{1-m}}$	$\frac{A}{1 + \beta e^{-Kt}}$	$Ae^{-Be^{-Kt}}$	$A(1 - Be^{-Kt})$	$A(1 - e^{-Kt})$
Upper asymptote	A	A	A	A	A	A
Starting point of the growth function	$A\{1 - B\}^d$	$\{A^{1-m} - b_1\}^{\frac{1}{1-m}}$	$\frac{A}{1 + B}$	Ae^{-B}	$A(1 - B)$	0
Growth rate	$ABKd(1 - Be^{-Kt})^{d-1} e^{-Kt}$	$\frac{b_1 K}{1 - m} e^{-Kt} y^m$	$\frac{ABK e^{-KT}}{(1 + Be^{-Kt})^2}$	$ABKe^{-Be^{-kt}} e^{-kt}$	$ABKe^{-kt}$	AKe^{-Kt}
Relative growth rate as function of time	$\frac{BKd}{e^{Kt} - B}$	$\frac{b_1 K}{(1 - m)(e^{Kt} A^{1-m} - b_1)}$	$\frac{BK}{e^{Kt} + B}$	$\frac{BKe^{-Be^{-kt}}}{e^{kt} - B}$	$\frac{BK}{e^{kt} - 1}$	$\frac{K}{e^{Kt} - 1}$
Relative growth rate as function of response variable	$Kd\left\{(A/y)^{\frac{1}{d}} - 1\right\}$	$\frac{K(A^{1-m} - y^{1-m})}{y^{(1-m)}(1 - m)}$	$K\left(1 - \frac{y}{A}\right)$	$\frac{Ky \ ln \frac{A}{y}}{A\left(1 - \ln\frac{A}{y}\right)}$	$\frac{BK(A - y)}{A(B - 1) + y}$	$\frac{K(A - y)}{y}$
Second derivative of the growth function	$ABK^2 d(Bde^{-Kt} - 1)(1 - Be^{-Kt})^{d-2} e^{-Kt}$	$\frac{b_1 K^2}{1 - m} e^{-Kt} \left\{-y^{1-m} + \frac{(y^{1-m} - A^{1-m}) y^{2m-1} m}{1 - m}\right\}$	$ABK^2 e^{-Kt}(1 + Be^{-Kt})^{-2}\{2Be^{-Kt}(1 - Be^{-Kt})^{-1} - 1\}$	$ABK^2 e^{-Be^{-kt}} e^{-kt}(-1 + e^{-kt})$	$-ABK^2 e^{-Kt}$	$-yAK^2 e^{-kt}$
Point of inflection $(y(t) =)$	$A\left(\frac{d - 1}{d}\right)^d$	$Ae^{\frac{\ln m}{1-m}}$	$\frac{A}{2}$	$\frac{A}{e}$	Does not exist	Does not exist
Domain of the independent variable	$[0, \infty)$	$[0, \infty)$	$[0, \infty)$	$[0, \infty)$	$[0, \infty)$	$[0, \infty)$
Domain of the dependent variable	$(A\{1 - B\}^d, A)$	$\left[\{A^{1-m} - b_1\}^{\frac{1}{1-m}}, A\right]$	$\left[\frac{A}{1 + B}, A\right]$	$[Ae^{-B}, A]$	$[A(1 - B), A]$	$[0, A]$

Table 5: Estimated Parameters and the Fitted Values of the Candidate Models along with the Summary of Statistical Analysis for Top Height Growth of Babul Tree in India

Age	Observed Data	Chapman Richards		Von Bertalanffy		Logistic						Gompertz						Monomolecular						Negative Exponential	
		A	B	A	B	A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F	A	B
5	8.14	8.14	8.13	8.14	8.21	8.14	8.14	8.19	8.11	8.15	8.17	8.14	8.14	8.18	8.10	8.54	8.18	8.14	8.14	8.14	8.16	8.14	8.14	8.14	7.83
10	12.19	12.19	12.15	12.19	12.21	11.86	12.19	11.91	12.33	12.07	12.04	12.03	12.19	12.05	12.26	11.78	12.09	12.21	12.19	12.19	12.18	12.19	12.19	12.83	12.47
15	14.93	14.93	14.86	14.93	14.92	14.93	14.93	14.98	15.27	15.10	15.08	14.93	14.93	14.94	15.11	14.58	14.96	14.93	14.93	14.91	14.89	14.91	14.91	15.52	15.21
20	16.70	16.81	16.72	16.76	16.73	16.91	16.28	16.95	16.73	16.88	16.90	16.83	16.53	16.82	16.77	16.79	16.80	16.76	16.78	16.74	16.73	16.74	16.74	17.08	16.84
25	17.98	18.11	18.01	17.98	17.94	17.98	16.84	18.01	17.35	17.75	17.81	17.98	17.40	17.96	17.68	18.43	17.90	17.98	18.04	17.96	17.98	17.96	17.96	17.98	17.81
Parameters	A	21.09	20.98	20.35	20.29	18.97	17.19	18.99	17.73	18.43	18.55	19.52	18.32	19.49	18.64	22.13	19.32	20.46	20.66	20.44	20.58	20.47	20.47	19.19	19.21
Parameters	B / b_1 / β	0.934	0.934	13.27	13.30	2.950	3.013	2.926	3.219	3.019	2.991	1.579	1.615	1.567	1.661	1.439	1.575	0.899	0.896	0.898	0.893	0.897	0.897	---	---
Parameters	k	0.358	0.357	0.418	0.416	0.796	0.997	0.796	0.997	0.873	0.855	0.591	0.689	0.591	0.689	0.413	0.606	0.400	0.391	0.400	0.391	0.398	0.398	0.551	0.523
Parameters	m / d	0.90	0.894	0.09	0.087	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---
χ^2		No χ^2 value at zero degree of freedom				0.012	0.088	0.011	0.032	0.008	0.007	0.003	0.021	0.003	0.008	0.053	0.002	.002	.006	.001	.002	.001	.001	0.063	0.026
RMSE		0.078	0.042	0.027	0.04	0.176	0.543	0.17	0.326	0.162	0.151	0.092	0.272	0.084	0.164	0.363	0..077	0.026	0.046	0.021	0.024	0.020	0.020	0.427	0.247
R^2 (in %)		99.95	99.98	99.99	99.98	99.75	97.61	99.76	99.13	99.78	99.81	99.93	99.40	99.94	99.78	98.92	99.95	99.99	99.98	99.99	99.99	99.99	99.99	98.51	99.51
R_a^2		0.99	0.99	0.99	0.99	0.99	0.90	0.99	0.96	0.99	0.99	0.99	0.97	0.99	0.99	0.95	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.98	0.99
$R^2_{prediction}$ (in %)		99.86	99.97	99.98	99.98	99.65	92.35	99.64	97.53	99.50	99.62	99.89	98.05	99.91	99.40	97.99	99.90	99.99	99.96	99.99	99.99	99.99	99.99	97.92	99.27

Table 6: 95% Confidence Intervals of the Parameters of the Candidate Models

Data	Models	Method	A		$B / b_1 / \beta$		K		m / d	
			Lower Limit	Upper Limit	Lower Limit	Upper Limit	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Top Height	Chapmen Richards	A	20.262	21.927	0.802	1.067	0.277	0.440	0.582	1.218
		B	20.496	21.466	0.857	1.012	0.309	0.405	0.709	1.078
	Von Bertalanffy	A	20.176	20.515	8.189	18.358	0.395	0.439	−0.014	0.196
		B	19.969	20.608	3.498	23.105	0.375	0.458	−0.116	0.290
	Logistic	A	15.695	22.243	1.229	4.671	0.312	1.281	−−−	−−−
		B	10.621	23.754	−4.168	10.195	−0.897	2.891	−−−	−−−
		C	15.827	22.167	1.269	4.583	0.325	1.268	−−−	−−−
		D	13.736	21.728	−1.185	7.624	−0.072	2.067	−−−	−−−
		E	15.926	20.929	1.224	4.814	0.396	1.350	−−−	−−−
		F	16.113	20.984	1.361	4.621	0.415	1.296	−−−	−−−
	Gompertz	A	17.240	21.798	1.223	1.937	0.357	0.825	−−−	−−−
		B	13.322	23.312	0.319	2.911	−0.081	1.459	−−−	−−−
		C	17.425	21.545	1.246	1.889	0.378	0.805	−−−	−−−
		D	15.612	21.674	0.879	2.444	0.241	1.137	−−−	−−−
		E	2.923	41.335	0.628	2.249	−0.438	1.263	−−−	−−−
		F	17.524	21.110	1.270	1.880	0.409	0.804	−−−	−−−
	Monomolecular	A	19.497	21.438	0.855	0.942	0.337	0.463	−−−	−−−
		B	18.895	22.427	0.823	0.969	0.282	0.499	−−−	−−−
		C	19.681	21.199	0.864	0.932	0.351	0.449	−−−	−−−
		D	19.656	21.511	0.854	0.931	0.333	0.448	−−−	−−−
		E	19.717	21.214	0.864	0.930	0.350	0.447	−−−	−−−
		F	19.719	21.216	0.864	0.930	0.350	0.446	−−−	−−−
	Negative Exponential	A	17.401	20.993	−−−	−−−	0.416	0.688	−−−	−−−
		B	18.090	20.322	−−−	−−−	0.446	0.601	−−−	−−−
Maximum Diameter	Chapmen Richards	A	41.403	44.831	0.979	1.074	0.255	0.361	0.736	1.064
		B	40.166	46.507	0.955	1.117	0.212	0.405	0.610	1.199
	Von Bertalanffy	A	38.936	41.719	−1.244	57.351	0.319	0.449	−0.156	0.338
		B	39.733	41.620	8.534	46.580	0.343	0.430	−0.064	0.262

Logistic	A	27.870	45.028	0.803	8.635	0.264	1.463	−−−	−−−
	B	11.976	50.416	−17.031	26.801	−2.070	4.354	−−−	−−−
	C	27.655	46.069	0.571	8.902	0.228	1.498	−−−	−−−
	D	21.738	44.134	−7.609	18.365	−0.541	2.825	−−−	−−−
	E	28.303	41.477	0.330	9.535	0.324	1.629	−−−	−−−
	F	28.894	41.433	0.782	8.993	0.369	1.548	−−−	−−−
Gompertz	A	31.863	44.417	1.480	2.648	0.326	0.860	−−−	−−−
	B	16.541	51.596	−1.221	5.515	−0.623	2.096	−−−	−−−
	C	32.782	43.719	1.548	2.559	0.360	0.826	−−−	−−−
	D	26.130	44.555	0.495	3.998	0.071	1.402	−−−	−−−
	E	−3.923	93.062	0.676	3.022	−0.457	1.289	−−−	−−−
	F	33.453	41.840	1.630	2.504	0.418	0.810	−−−	−−−
Monomolecular	A	36.172	47.168	0.907	1.093	0.227	0.465	−−−	−−−
	B	35.563	47.366	0.899	1.104	0.219	0.480	−−−	−−−
	C	37.346	46.261	0.927	1.077	0.250	0.442	−−−	−−−
	D	37.286	46.005	0.928	1.080	0.254	0.446	−−−	−−−
	E	37.351	46.307	0.926	1.077	0.250	0.442	−−−	−−−
	F	37.328	46.101	0.928	1.079	0.253	0.444	−−−	−−−
Negative Exponential	A	40.692	42.669	−−−	−−−	0.330	0.362	−−−	−−−
	B	40.796	42.529	−−−	−−−	0.333	0.361	−−−	−−−

Table 7: Estimated Parameters and the Fitted Values of the Candidate Models along with the Summary of Statistical Analysis for Maximum Diameter Growth of Babul Tree in India

Age	Observed Data	Chapman Richards		Von Bertalanffy		Logistic						Gompertz						Monomolecular						Negative Exponential	
		A	B	A	B	A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F	A	B
5	12.19	12.19	11.89	12.19	12.07	12.19	12.19	12.30	12.12	12.21	12.23	12.19	12.19	12.29	12.06	13.17	12.30	12.19	12.19	12.18	12.17	12.18	12.17	12.19	12.22

10	20.83	20.83	20.68	20.83	20.86	19.82	20.83	20.01	21.28	20.52	20.45	20.31	20.83	20.43	21.12	19.94	20.55	20.82	20.83	20.85	20.87	20.85	20.86	20.83	20.86
15	26.92	26.92	26.87	26.92	27.06	26.92	26.92	27.20	28.03	27.61	27.56	26.92	26.92	27.04	27.62	26.23	27.13	26.92	26.92	26.98	27.00	26.98	26.99	26.92	26.96
20	31.49	31.30	31.33	31.14	31.36	31.71	29.69	32.06	31.19	31.74	31.80	31.46	30.44	31.58	31.41	31.42	31.53	31.24	31.21	31.32	31.32	31.32	31.32	31.49	31.27
25	34.29	34.47	34.56	34.05	34.32	34.29	30.70	34.67	32.36	33.63	33.79	34.29	32.28	34.41	33.40	35.39	34.20	34.29	34.24	34.39	34.37	34.39	34.38	34.29	34.32
Parameters	A	43.12	43.34	40.33	40.68	36.45	31.19	36.86	32.94	34.89	35.16	38.14	34.07	38.25	35.34	44.57	37.65	41.67	41.46	41.80	41.65	41.83	41.72	41.68	41.66
	B / b_1 / β	0.026	1.036	28.054	27.557	4.719	4.885	4.737	5.378	4.933	4.888	2.064	2.147	2.053	2.247	1.849	2.067	1.000	1.002	1.002	1.004	1.002	1.003	---	---
	k	0.308	0.309	0.384	0.386	0.863	1.142	0.8634	1.142	0.976	0.958	0.593	0.737	0.593	0.737	0.416	0.614	0.346	0.349	0.346	0.349	0.3458	0.348	0.346	0.347
	m / d	0.900	0.905	0.091	0.099	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---
χ^2		No χ^2 value at zero degree of freedom				0.053	0.529	0.052	0.172	0.037	0.032	0.014	0.162	0.010	0.047	0.165	0.007	0.002	0.003	0.001	0.001	0.001	0.001	0.002	0.001
$RMSE$		0.118	0.206	0.189	0.101	0.464	1.796	0.497	1.026	0.461	0.423	0.235	1.015	0.205	0.528	0.829	0.170	0.113	0.126	0.093	0.092	0.093	0.092	0.114	0.100
R^2 (in %)		99.98	99.93	99.94	99.98	99.66	94.85	99.60	98.32	99.66	99.71	99.91	98.36	99.93	99.55	98.90	99.95	99.98	99.98	99.99	99.99	99.99	99.99	99.98	99.98
R_a^2		0.99	0.99	0.99	0.99	0.98	0.79	0.98	0.93	0.98	0.98	0.99	0.93	0.99	0.98	0.96	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
$R^2_{prediction}$ (		99.94	99.87	99.86	99.97	99.58	84.12	99.35	95.31	99.24	99.43	99.89	94.96	99.91	98.89	97.87	99.93	99.96	99.95	99.97	99.97	99.97	99.97	99.96	99.96

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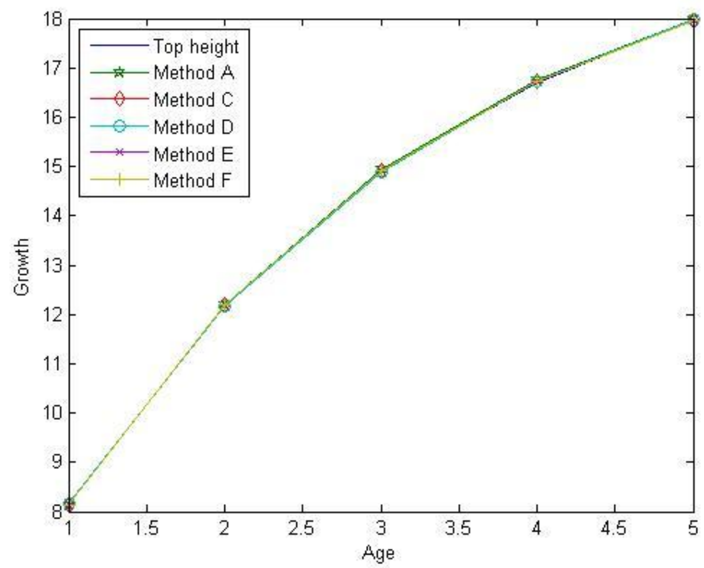

Figure 1: Observe and Estimated Values for Top Height Growth of Babul Tree Using Monomolecular Growth Model with Method A, C, D, E and F

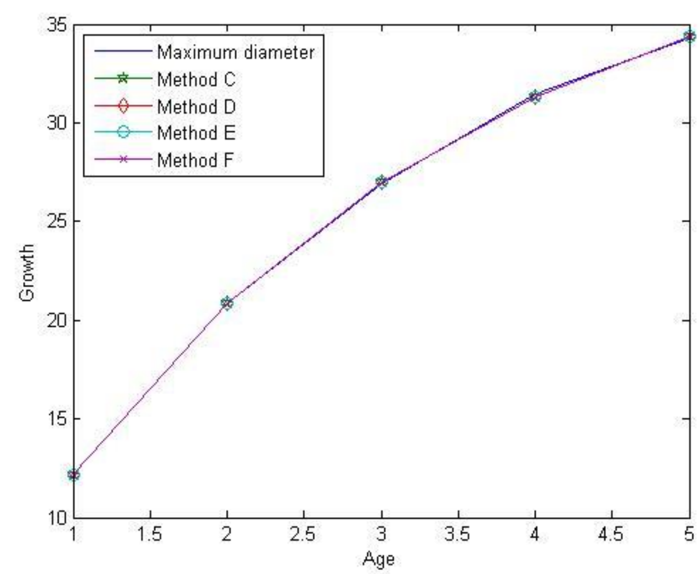

Figure 2: Observe and Estimated Values for Maximum Diameter Growth of Babul Tree Using Monomolecular Growth Model with Method C, D, E and F

Conclusion

This paper attempted to find the best fit growth model along with the best method of estimation for the babul growth in India based on the available data. This study presents a comparative study among Chapmen Richards's growth models and its various limiting cases. By observing all the results and discussion, it can be concluded that, the Monomolecular growth model was more reasonable over the remaining growth models for describing the growth of babul in India. One may consider any method of estimation (From method A to method F) to estimate the parameters of the Monomolecular growth model.

ACKNOWLEDGMENT
We would like to convey my heartfelt thanks to Mrs. Pallabi Saikia Mahanta, Assistant Professor, School of Basic Science, Assam Kaziranga University for her help and support. We would also like to convey our gratitude to our friends for providing us with resources which were indispensable for the success of this work.

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