**INTEGRAL SOLUTIONS OF** \( x^3 + y^3 + z^3 = 3xyz + 14(x + y)w^3 \)

**Gopalan M.A., **Malli S. and Vidhyalakshmi S.

*Department of Mathematics, Shrimathi Indira Gandhi College, Trichy-620002*

*Author for Correspondence*

**ABSTRACT**

The non-homogeneous quadratic equation with four unknowns represented by \( x^3 + y^3 + z^3 = 3xyz + 14(x + y)w^3 \) is analyzed for finding its non-zero distinct integral solutions. Four different methods have been presented for determining the integral solutions of the non-homogeneous bi-quadratic equation under consideration. Employing the integral solutions of the above equation, a few interesting relations between special numbers are exhibited.

**Keywords:** Bi-quadratic Equation with Four Unknowns, Integral Solutions

**MSc 2000 Mathematics Subject Classification:** 11D25

**INTRODUCTION**

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, bi-quadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity (Dickson, 1952; Mordell, 1969; Carmichael, 1959; Telang, 1996; Smart, 1999). In this context, one may refer Gopalan and Pandichelvi (2009); Gopalan and Shanmuganandham (2010); Gopalan and Padma (2010); Gopalan and Sangeetha (2011); Manju et al., (2011); Manju et al. (2012); Gopalan and Shanmuganandham (2012); Meena et al., (2014); Gopalan et al., (2014) for various problems on the bi-quadratic diophantine equations with four variables. However, often we come across non-homogeneous bi-quadratic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous equation with four unknowns given by \( x^3 + y^3 + z^3 = 3xyz + 14(x + y)w^3 \). A few relations among the solutions are presented.

**Notations**

\[
t_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right).
\]

\[
P_n^m = \frac{n(n+1)}{6}[(m-2)n+(5-m)]
\]

\[
Pt_n = \frac{n(n+1)(n+2)(n+3)}{24}
\]

\[
SO_n = n(2n^2 - 1)
\]

\[
S_n = 6n(n-1) + 1
\]

\[
Pr_n = n(n+1)
\]

**Method of Analysis**

The bi-quadratic Diophantine equation with four unknowns to be solved for its non-zero distinct integral solution is

\[
x^3 + y^3 + z^3 = 3xyz + 14(x + y)w^3
\]

Different patterns of integer solutions to (1) are illustrated below:

**Pattern I**

Introducing the linear transformations
Research Article

\[ x = u + v, \quad y = u - v, \quad z = 2u \]  \hspace{1cm} (2)

in (1), it is written as

\[ u^2 + 3v^2 = 7w^3 \]  \hspace{1cm} (3)

write 7 as

\[ 7 = (2 + i\sqrt{3})(2 - i\sqrt{3}) \]  \hspace{1cm} (4)

Assume

\[ w = a^2 + 3b^2 \]  \hspace{1cm} (5)

where a and b are non-zero distinct integers.

Using (4) and (5) in (3) and employing the method of factorization, define

\[ u + i\sqrt{3}v = (2 + i\sqrt{3})(a + i\sqrt{3}b)^3 \]

Equating real and imaginary parts in the above equation, we get,

\[ u = 2a^3 - 18ab^2 - 9a^2b + 9b^3 \]
\[ v = a^3 - 9ab^2 + 6a^2b - 6b^3 \]

Substituting the values of u, v in (2), we have

\[ x(a,b) = 3a^3 - 27ab^2 - 3a^2b + 3b^3 \]
\[ y(a,b) = a^3 - 9ab^2 - 15a^2b + 15b^3 \]
\[ z(a,b) = 4a^3 - 36ab^2 - 18a^2b + 18b^3 \]  \hspace{1cm} (6)

Thus (5) and (6) represent the non-zero distinct integer solutions of (1).

**Properties**

- \[ 2x(a+1,a) + 2y(a+1,a) + 108t_{4,a} + 24t_{3,a} - 8 \] is a cubical integer.
- \[ z(a,1) + w(a,1) - 8P_a^5 + 42t_{3,a} \equiv 0 \pmod{3} \]
- \[ w(1,2^n) - 3Ky_n \] is divisible by 2.
- \[ y(a,1) + z(a,1) - 10P_a^5 + 6S_a + 24t_{4,a} \] is divisible by 3.
- \[ x(1,b) + w(1,b) - 6P_b^5 + 6t_{3,b} + 24t_{4,b} = 1 \]
- \[ z(2^n,1) - 4y(2^n,1) - 42Ky_n \equiv 0 \pmod{2} \]
- \[ z(a,1) - 14y(a,1) - 18w(a,1) + 45 \] is a nasty number.
- \[ x(1,b) - 3y(1,b) + 21S_b = 0 \pmod{21} \]
- \[ 7[z(1,b) - 4y(1,b) + 63OH_b] \] is a perfect square.

**Pattern II**

(3) can be written as

\[ u^2 + 3v^2 = 7*1*w^3 \]  \hspace{1cm} (7)

Write 1 as

\[ 1 = \frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} \]  \hspace{1cm} (8)

Substituting (4), (5) and (8) in (7), proceeding as in Pattern I, the non-zero distinct integral solutions to (1) are given by

\[ x(a,b) = a^3 - 9ab^2 - 15a^2b + 15b^3 \]
Research Article

\[ y(a,b) = -2a^3 + 18ab^2 - 12a^2b + 12b^3 \]
\[ z(a,b) = -a^3 + 9ab^2 - 27a^2b + 27b^3 \]
\[ w(a,b) = a^2 + 3b^2 \]

**Properties**

- \[ 2x(a,a+1) + y(a,a+1) - 84P_{ra} - 84t_{3,a} + 42t_{4,a} = 42 \]
- \[ y(b+1,b) - 2z(b+1,b) - 42P_{rb} - 42t_{4,b} = 0 \]
- \[ y(1,b) - x(1,b) - 3z(1,b) - 42P_{rb} + 42t_{4,b} + 42S_{ob} = 0 \]
- \[ y(a,1) - 2z(a,1) + 12w(a,1) + 6 \text{ is a nasty number.} \]
- \[ x(2^n,1) + z(2^n,1) + w(2^n,1) + 123J_{2n} = 4 \]

**Pattern III**

Using the linear transformations

\[ x = 3(\alpha + \beta), y = 3(\alpha - \beta), z = 6\alpha, w = 3\gamma \]  

in (1), it leads to

\[ \alpha^2 + 3\beta^2 = 21\gamma^3 \]

write 21 as

\[ 21 = (3 + i2\sqrt{3})(3 - i2\sqrt{3}) \]

Assume

\[ \gamma = R^2 + 3S^2 \]

substituting (11) and (12) in (10) and employing the method of factorization, define

\[ \alpha + i\sqrt{3}\beta = (3 + i2\sqrt{3})(R + iS\sqrt{3})^3 \]

Equating the real and imaginary parts, we get

\[ \alpha = 3R^3 - 27RS^2 - 18R^2S + 18S^3 \]
\[ \beta = 2R^3 - 18RS^2 + 9R^2S - 9S^3 \]

Using these values of \( \alpha \) and \( \beta \) in (9), the non-zero distinct integral solutions to (1) are obtained as

\[ x(R,S) = 15R^3 - 135RS^2 - 27R^2S + 27S^3 \]
\[ y(R,S) = 3R^3 - 27RS^2 - 81R^2S + 81S^3 \]
\[ z(R,S) = 18R^3 - 162RS^2 - 108R^2S + 108S^3 \]
\[ w(R,S) = 3R^2 + 9S^2 \]

**Properties**

- \( x(R,1) - 5y(R,1) + 21w(R,1) + 189 \text{ is a perfect square.} \)
- \( x(2^n,1) - 5y(2^n,1) + w(2^n,1) - 1143J_{2n} = 12 \)
- \( z(R,1) - 6y(R,1) + 36w(R,1) + 54 \text{ is a nasty number} \)
- \( 6y(1,S) - z(1,S) + 42w(1,s) - 756P_S^5 + 378P_SrS - 378t_{4,S} = 126 \)
- \( y(R,1) + w(R,1) - 6P_r^5 + 162t_{3,r} \equiv 0 \text{ (mod 3)} \)

**Pattern IV**

In (11), 21 can also be written as
\[ 21 = \frac{(3 + 5i\sqrt{3})(3 - 5i\sqrt{3})}{4} \]  

(13)

Using (13) and (12) in (11) and proceeding as in pattern III, we get the non-zero distinct integral solutions to (1) be

\[
\begin{align*}
x(R, S) &= 12R^3 - 108RS^2 - 54R^2S + 54S^3 \\
y(R, S) &= -3R^3 + 27RS^2 - 81R^2S + 81S^3 \\
z(R, S) &= 9R^3 - 81RS^2 - 135R^2S + 135S^3 \\
w(R, S) &= 3R^2 + 9S^2
\end{align*}
\]

(14)

**Properties**

- \(x(R,1) + 4y(R,1) + 126w(R,1) + 24\) is a nasty number.
- \(z(R,1) + 3y(R,1) - 72w(R,1) + 216t_{3,R}\) is an even number.
- \(9x(1,S) - 12z(1,S) - 1134P_{rS} + 2268P^5_S = 0\)
- \(z(1,S) + 3y(1,S) + 42w(1,S) - 756P^5_S + 378P_{rS} - 378t_{4,S} = 126.\)
- \(z(2^n,1) + 3y(2^n,1) + 378j_{2n} \equiv 0 (\text{mod} 2)\)

**CONCLUSION**

In this paper, we have presented four different patterns of integer solutions to the bi-quadratic with four unknowns given by \(x^3 + y^3 + z^3 = 3xyz + 14(x + y)w^3\). It is to be noted that, in addition to (4) in pattern I and (8) in pattern II, we have the following choices:

\[
\begin{align*}
7 &= \frac{(1 + i3\sqrt{3})(1 - i3\sqrt{3})}{4} \\
1 &= \frac{(1 + 4i\sqrt{3})(1 - 4i\sqrt{3})}{49} \\
\text{and} \quad 1 &= \frac{(1 + 15i\sqrt{3})(1 - 15i\sqrt{3})}{26}
\end{align*}
\]

Following the procedure presented in this paper, one may obtain other choices of integer solutions to the equation under consideration. To conclude, one may search for the integral solutions of the bi-quadratic equation with four or more variables.

**REFERENCES**


Gopalan MA and Sangeetha G (2010). Integral solutions of Non-homogeneous Quartic equation \(x^4 - y^4 = (2\alpha^2 + 2\alpha + 1)(z^2 - w^2)\) Impact Journal of Science and Technology 4(3) 15-21.
Research Article

\[ x^4 + x^2 + y^2 - y = z^2 + z, \]

Gopalan MA and Shanmuganandham P (2010). On the biquadratic equation \( x^4 + y^4 + z^4 = 2w^4 \),

Gopalan MA and Shanmuganandham P (2012). On the biquadratic equation
\[ x^4 + y^4 + (x + y)z^3 = 2(k^2 + 3)2^n w^4, \]

Gopalan MA, Usha Rani TR and Vidhyalakshmi S (2014). Homogeneous equation with four
unknowns \((x + y)(x^3 + y^3) = 52z^2w^2\)

Quartic equation \( z^4 - y^4 = (k^2 + 1)(z^2 - w^2) \)

Manju Somanath, Sangeetha G and Gopalan MA (2012). Integral solutions of the bi-quadratic
equation with four unknowns given by \( xy + (k^2 + 1)z^2 = 5w^4 \)

equation \( x^3 + y^3 = 39zw^3 \)
*IJOER* **2**(1) 57-60.


Smart ND (No Date). *The Algorithmic Resolutions of Diophantine Equations* (Cambridge University