

RELATIONSHIP BETWEEN THE LABELLED ORIENTED GRAPH GROUP AND THE CYCLICALLY PRESENTED GROUP

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ABSTRACT

The aim of present work was to study a class of Labelled Oriented Graph (LOG) group for which the underlying graph is a circuit with vertex set V and edge set E . The relationship between the LOG group and the cyclically presented group was investigated. The obtained results indicated the hyperbolicity of the cyclically presented group implied the solvability of the conjugacy problem for the LOG group. In the vertices in the circuit, the LOGs and cyclically presented groups coincided by the generalized Fibonacci groups. The small cancellation and curvature methods have been used for hyperbolicity, automaticity, and solvability of decision problems.

Keywords: LOG Groups, Cyclically Presented Group, Fibonacci Group, Hyperbolic Group

INTRODUCTION

A Labelled Oriented Graph (LOG) consists of a finite connected graph equipped with loops, vertexes and multiple edges (Szczepan and Vesnin, 2001; Gilbert and Howie, 1995; Howie and Williams, 2012). The vertex set is V and E is the edge set. The initial vertex map, terminal vertex map and labelling map are defined as ι , τ and λ . The LOG determines a corresponding LOG presentation as follows (Szczepan and Vesnin, 2001):

$$K(\Gamma) = \langle V\Gamma \mid \tau(e)^{-1}\iota(e)\lambda(e), e \in E\Gamma \rangle \quad (1)$$

The Fibonacci group $G_n(m, k)$ is defined as follows (Bardakov and Vesnin, 2003; Cavicchioli *et al.*, 2008; Edjvet and Howie, 2008):

$$G_n(m, k) = \langle x_1, \dots, x_n : x_i x_{i+m} = x_{i+k} \ (i = 1, \dots, n) \rangle \quad (2)$$

Where $0 < m < k < n, (n, m, k) = 1$.

The cyclically presented group $G_n(W)$ is defined as follows (Johnson *et al.*, 1999; Sela, 1999; Bogopolski *et al.*, 2010; Kasaeipoor *et al.*, 2015; Maghsoudi *et al.*, 2012; Dahmani and Guirardel, 2011; Cavicchioli *et al.*, 1998):

$$G_n(w) = \langle x_1, \dots, x_n \mid w, \theta(w), \dots, \theta^{n-1}(w) \rangle \quad (3)$$

The natural extension of $G_n(w)$ is defined as follows:

$$G'_n(w) = \left[G_n(w), t \mid t^{-1}gt = \phi(g), g \in G_n(w) \right] \quad (4)$$

Theorem 1: $G'_n(w)$ is defined as follows:

$$G'_n(w) = \left[a, c \mid W(a, c), (a, c^n) \right], W(a, c) = w(ac^{-1}, cac^{-2}, \dots, c^{n-1}ac^{-n}) \quad (5)$$

Proof 1: By applying $K(n, t)$ for equivalence complex:

$$\begin{aligned} K &\cong \langle a, b, c \mid a^{-1}ca = b, c^{1-t}bc^{t-1} = a, c^{t-n-1}ac^{n-t+1} = b \rangle \cong \langle a, b, c \mid a^{-1}ca = b, c^{1-t}bc^{t-1} = a, c^{-n}ac^n = a \rangle \\ &\cong \langle a, b, c \mid a^{-1}ca = b, c^{1-t}a^{-1}c^{t-1} = a, c^{-n}ac^n = a \rangle \cong \langle a, c \mid c^{1-t}a^{-1}cac^{t-1} = a, c^{-n}ac^n = a \rangle. \end{aligned} \quad (6)$$

The Groups

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The LOG presentation is given as follows:

$$\left\langle \left(\prod_{i=0}^{r-1} ac^{p_{i+1}-p_i} \right)^{-1} c \left(\prod_{i=0}^{r-1} ac^{p_{i+1}-p_i} \right) a^{-1}, |a, c^n| \right\rangle \quad (7)$$

By consideration the isomorphic to the natural HNN extension:

$$G_n((x_{p_0} x_{p_1} + 1 \dots x_{p_{r-1}} + (r-1)x_{p_r+r})(x_{p_0+1} x_{p_1+2} \dots x_{p_{r-1}+r})^{-1}) \quad (8)$$

The presentation of the LOG group is given as follows:

$$w(x_1, \dots, x_n) = (x_{p_0} x_{p_1+1} \dots (r-1)x_{p_r+r})(x_{p_0+1} \dots x_{p_{r-1}+r})^{-1} \quad (9)$$

where

$$w'(x_1, \dots, x_n) = x_{p_0} x_{p_1+1} + x_{p_{r-1}} + (r-1) \quad (10)$$

Therefore:

$$\begin{aligned} w(ac^{-1}, cac^{-2}, c^2 ac^{-3}, \dots, c^{n-1} ac^{-n}) &= w'(ac^{-1}, cac^{-2}, c^2 ac^{-3}, \dots, c^{n-1} ac^{-n}) c^{p_r+r} ac^{-(p_r+r)} \\ w'(cac^{-2}, c^2 ac^{-3}, c^3 ac^{-4}, \dots, c^n ac^{-(n+1)})^{-1} &= w'(ac^{-1}, cac^{-2}, c^2 ac^{-3}, \dots, c^{n-1} ac^{-n}) c^{p_r+r-1} ac^{-(p_r+r-1)} \\ w'(ac^{-1}, cac^{-2}, c^2 ac^{-3}, \dots, c^{n-1} ac^{-n})^{-1} c^{-1} & \end{aligned} \quad (11)$$

Cyclically Presented Groups

By consideration the cyclically presented group in the form of:

$$\langle a, c | U(a, c), [a, c^n] \rangle \quad (12)$$

$$w(x_1, \dots, x_n) = x_1^{\alpha_1} x_{1+\gamma_1}^{\alpha_2} \dots x_{1+\gamma_1+\dots+\gamma_{k-1}}^{\alpha_k} \quad (13)$$

Lemma 1: The n, t are the non-negative integers that $n > t$. For $0 < s < n$ and $t \equiv (t-1)s$ modulo n . The groups of $G(n, t)$ and $G(n, s)$ are isomorphic.

For $n > 1$, the group $G(n, 1)$ is trivial. For $n > 0$, the group $G(b, 0)$ is cyclic of order $2^n - 1$. For $\alpha > 0$, the group $(2k, k+1)$ is cyclic of order 2^{k+1} .

$$G(2t-1, t) \equiv G(2t-1, 2t-2) \equiv F(2, 2t-1) \quad (14)$$

$H_2(X)$ is infinite cyclic, generated by the class of the relator $[a, c^n]$. $H_1(X)$ is infinite cyclic generated by the class of the generator c . In the group $a, c | [a, c^n]$, the element c^n is central and $[U, c^n] = 1$ is a relator. Moreover, $[a, c^n]^{-1}$ was more than $[a, c^n]^{-1}$.

For the circuit sample with vertex set V and edge set E , the $\phi: \square \pi_1(x) \rightarrow \pi_1(x)$ is the map of $\square \pi_1(x)$.

The subgroup of homotopy group is given as follows:

$$\langle x_j (j \in \square) | x_{j+n}^{-1} (j \in \square), w_1, \dots, w_n \rangle \quad (15)$$

By consideration the $s: \pi_2(Z) \rightarrow \pi_1(X)$ of $Z\pi_1(x)$, the Hurewicz isomorphism theorem implies that

$$\pi_2(X) = H_2(X) \subseteq C_2(x). \quad (16)$$

The Conjugacy Problem

The conjugacy problem in the natural HNN extension $G'_n(w) = G_n(w) \times \square$ via the twisted conjugacy problem in $G_n(w)$.

Theorem: G is a finitely generated hyperbolic group and $\phi \in \text{Aut}(G)$ has finite order. The ϕ -twisted conjugacy problem in G is solvable. The conjugacy problem is solvable for hyperbolic groups. Φ has finite order and G is finitely generated. δ is the hyperbolicity constant for the and geodesic quadrilaterals are 2δ -slim. By using ϕ to each edge of γ gives a geodesic segment $\phi(\gamma)$ from 1 to $\phi(g)$. G is hyperbolic and $\phi \in \text{Aut}(G)$ has finite order then the conjugacy problem for $G \rtimes_\phi \mathbb{Z}$ is solvable. The cyclically presented group $G_n(w)$ is hyperbolic and the conjugacy problem for HNN extension $G_n(w)$ is solvable.

Proof: t is a generator of \mathbb{Z} which $G \rtimes_\phi \mathbb{Z}$ has elements gt^m with $g \in G, m \in \mathbb{Z}$. Two elements $ut^p, vt^q \in G \rtimes_\phi \mathbb{Z}$ are conjugate. $gt^m \in G \rtimes_\phi \mathbb{Z}$ which $(ut^p)(gt^m) = (gt^m)(vt^q)$; $g \in G, m \in \mathbb{Z}$ which $(u\phi^p(g))t^{p+m} = (g\phi^m(v))t^{m+q}$; $g \in G, m \in \mathbb{Z}$. $u\phi^p(g) = g\phi^m(v)$; $m \in \mathbb{Z}$ which $u \sim \phi^p \phi^m(v)$. The conjugacy problem is solvable in $G \rtimes_\phi \mathbb{Z}$.

Cancellation Conditions

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By consideration the small cancellation conditions for the presentations $H_n(m, k)$, the strongest $C(p)$ condition that $H_n(m, k)$ can satisfy is $C(4)$. The prepared condition is considered for triangular presentations.

The group $H_n(m, k)$ has solvable and conjugacy problems and is automatic. $H_n(m, k)$ is non-elementary hyperbolic and the class of groups $H_n(m, k)$ has solvable isomorphism problem. Moreover, the natural HNN $H_n(m, k)$ has solvable conjugacy problem. Therefore, $H_n(m, k)$ acts freely on a finite-dimensional contractible complex.

Proof. In order to prove that $H(n, 3)$ is hyperbolic. It is necessary to indicate that it has a linear isoperimetric function as $f: \mathbb{N} \rightarrow \mathbb{N}$ which for all $N \in \mathbb{N}$ and all freely reduced words $w \in F_n$ with length at most N that represent the identity of $H(n, 3)$. The minimum number of 2-cells in a reduced via the presented diagram over $H(n, 3)$ with boundary of w . The boundary of diagram is a simple closed curve. Note that each 2-cell in a diagram over $H(n, 3)$ is a triangle. The corners of vertices each have angle 70° .

CONCLUSION

In the present study, hyperbolicity of the cyclically presented group implies the solvability of the conjugacy problem for the LOG group. The small cancellation and curvature methods have been used to obtain results on hyperbolicity, and solvability of problems. The obtained results are in good agreement with Fibonacci group that is cyclically present group with infinite structure.

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