Indian Journal of Fundamental and Applied Life Sciences ISSN: 2231– 6345 (Online)
An Open Access, Online International Journal Available at www.cibtech.org/sp.ed/jls/2015/04/jls.htm
2015 Vol. 5 (S4), pp. 1256-1265/Akrami and Hosseini
Research Article

A QUADRATIC FUZZY PENALTY METHOD FOR SOLVING A CLASS OF NONLINEAR FUZZY OPTIMIZATION PROBLEMS

***Abbas Akrami[1], Mohammad Mehdi Hosseini[2]**
[1]*Department of Applied Mathematics, Faculty of Mathematics, Yazd University, Yazd, Iran.*
[2]*Department of Applied Mathematics, Faculty of Mathematics and Computer Science, ShahidBahonar University of Kerman, Kerman, Iran.*
**Author for Correspondence*

ABSTRACT
Some important methods for solving constrained fuzzy optimization problems replace the original problem by a sequence of sub-problems in which the constraints are represented by terms added to the objective function. In this paper, we describe one approach of this type for solving a class of nonlinear fuzzy optimization problems. The fuzzy quadratic penalty method adds a multiple of the square of the violation of each constraint to the objective function. The penalty terms for the constraint violations are multiplied by a positive coefficient. By making this coefficient larger, we penalize constraint violations more severely, thereby forcing the minimizer of the penalty function closer to the feasible region for the constrained problem. We described the quadratic fuzzy penalty method and investigate convergence of this method.

Keywords: *Fuzzy Penalty Method, Fuzzy Nonlinear Optimization, Fuzzy Triangular Numbers.*

INTRODUCTION
In real programming, the coefficients of the problems are evermore treated as deterministic values. However, ambiguity evermore exits in practical engineering problems. Therefore, fuzzy and stochastic approaches are generally used to illustrate the imprecise characteristics. In stochastic optimization the uncertain coefficients are regarded as random variables and their probability distributions are assumed to be known *e.g.* (Charnes and Cooper**,** 1959; Kall,1982; Liu *et al.*, 2003; Cho and Gyeong**,** 2005). In fuzzy optimization the constraints and objective functions are viewed as fuzzy sets and their membership functions need to be known. E.g. (Slowinski, 1986; Delgado *et al.*, 1989; Luhandjula, 1989; Liu and Iwamura, 2001). In these methods, the membership functions and probability distributions play important roles. However, it is often difficult to specify an appropriate membership function or accurate probability distribution in an unclear environment.
Fuzzy set theory has been applied to many disciplines such as control theory and operational research, mathematical modeling and industrial applications. Tanaka, *et al.*, (1974) first proposed the concept of fuzzy optimization on general level. Zimmerman, (1978) proposed the first formatting of fuzzy linear programming. Kumar and Kaur, (2010) and Kheirfam, (2011) introduced an optimal solution of fuzzy nonlinear programming problems. In their works, they have taken all coefficients and decision variables to be fuzzy numbers and all the constraints to be linear. Behara and Nayak, (2012) have developed KKT conditions for solving fuzzy nonlinear programming problems with continuous and differentiable objective function and constraints. Jamison and Lodwic, (2001) have solved fuzzy linear programming using a penalty method. Nevertheless, they used membership degrees as levels of possibility. Penalty function method for solving fuzzy nonlinear programming problem was proposed by Jameel and Radhi, (2014). In their work, the penalty function method has been developed and mixed with Nelder and Mend's algorithm of direct optimization problem. Solution have been used together to solve fuzzy nonlinear programming problem.
In this paper, we focus on solving fuzzy nonlinear optimization problems by using the quadratic fuzzy penalty method. We take all coefficients of the objective function and constraints to be triangular fuzzy numbers. The quadratic fuzzy penalty method adds a multiple of the square of the violation of each constraint to the objective function. Because of its simplicity and intuitive appeal, this approach is used often in practice.

This paper is organized as follows: in section 2, some basic definitions and arithmetic operations of triangular fuzzy numbers are reviewed. In addition, fuzzy-value function with its continuity and differentiability is described. In section 3, formulations of fuzzy nonlinear programming problems are discussed. In section 5, the quadratic fuzzy penalty method is proposed. In section 6, the convergence of this approach is described. In section 7, to demonstrate the effectiveness of the proposed method, some examples are solved. The conclusion appears in last section.

Preliminaries

Definition 2.1 (Pathak and Pirzada, 2011) Suppose that R be the set of real numbers and $\tilde{a}: R \to [0,1]$be a fuzzy set. If $\tilde{a}$ satisfies the following properties, then we say that $\tilde{a}$is a fuzzy number.

(i) $\tilde{a}$is normal, that is, there exists $x_0 \in R$ such that $\tilde{a}(x_0) = 1$.

(ii) $\tilde{a}$is fuzzy convex, that is,

$\tilde{a}(tx + (1-t)y) \geq min\{\tilde{a}(x), \tilde{a}(y)\}$; $\forall\, x, y \in R, t \in [0,1]$.

(iii) $\tilde{a}$ is upper semi continuous on R, that is,$\{x|\tilde{a}(x) \geq \alpha\}$ is a closed subset of R for each $\alpha \in [0,1]$.

(iv) $cl\{x \in R|\tilde{a}(x) > 0\}$Forms a compact set.

For all$\alpha \in (0,1]$, α-level set,$\tilde{a}_\alpha$, of any $\tilde{a} \in F(R)$is defined as $\tilde{a}_\alpha = \{x \in R|\tilde{a}(x) \geq \alpha\}$. Where, F(R) denotes the set of all fuzzy numbers on R. The 0-level set $\tilde{a}_0$ is defined as the closure of the set$\{x \in R|\tilde{a}(x) > 0\}$.

By definition of fuzzy numbers, it has been proved that, for any$\tilde{a} \in F(R)$ and for each $\alpha \in (0,1]$, $\tilde{a}_\alpha$is compact convex subset of R, and we write$\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$.

Definition 2.2 (Jamison and Lodwick, 2001) According to Zadeh's extension principle, addition and scalar multiplications of two fuzzy numbers $\tilde{a}$and $\tilde{b}$by their α-cuts are defined as follows:

$(\tilde{a} \oplus \tilde{b})_\alpha = [\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha]$,

$(\mu \otimes \tilde{a})_\alpha = [\mu\tilde{a}^L_\alpha, \mu\tilde{a}^U_\alpha]$.

Definition 2.3 (Pathak and Pirzada, 2011) Suppose that$\tilde{a}$ *and* $\tilde{b}$ are two fuzzy numbers. If the fuzzy number $\tilde{c}$ exists such that $\tilde{c} \oplus \tilde{b} = \tilde{a}$ then we say that $\tilde{c}$ is Hukuhara difference of $\tilde{a}, \tilde{b}$ and denoted by $\tilde{c} = \tilde{a} \ominus_H \tilde{b}$.

We define difference of $\tilde{a}$ *and* $\tilde{b}$by their α-cuts by use of H-difference as follows:

$(\tilde{a} - \tilde{b})_\alpha$=$\tilde{a}_\alpha \ominus_H \tilde{b}_\alpha = [\tilde{a}^L_\alpha - \tilde{b}^L_\alpha, \tilde{a}^U_\alpha - \tilde{b}^U_\alpha]$,

where$\tilde{a}, \tilde{b} \in F(R), \mu \in R$ and $\alpha \in [0,1]$.

Proposition 2.4 (Pathak and Pirzada, 2011) Let$\tilde{a} \in F(R)$, we have

(i) $\tilde{a}^L_\alpha$ is bounded left continuous nondecreasing function on (0,1];

(ii) $\tilde{a}^U_\alpha$ is bounded left continuous nonincreasing function on (0,1];

(iii) $\tilde{a}^L_\alpha$ and $\tilde{a}^U_\alpha$are right continuous at $\alpha = 0$;

(iv) $\tilde{a}^L_\alpha \leq \tilde{a}^U_\alpha$.

In addition, if the pair of functions $\tilde{a}^L_\alpha$and $\tilde{a}^U_\alpha$ satisfy the conditions (i)-(iv), then there exists a unique$\tilde{a} \in F(R)$ such that for all $\alpha \in [0,1]$ we have $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$.

Here, we define a partial order relation on fuzzy number space.

Definition 2.5 (Pathak and Pirzada, 2011) Let $\tilde{a}, \tilde{b} \in F(R)$and $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$, $\tilde{b}_\alpha = [\tilde{b}^L_\alpha, \tilde{b}^U_\alpha]$ be two closed intervals in R, for all $\alpha \in [0,1]$, we define

(i) $\tilde{a} \preceq \tilde{b} \Leftrightarrow \tilde{a}^L_\alpha \leq \tilde{b}^L_\alpha,\ \tilde{a}^U_\alpha \leq \tilde{b}^U_\alpha$.

(ii) $\tilde{a} \prec \tilde{b}$ if and only if for all $\alpha \in [0,1]$:

$$\begin{cases}\tilde{a}^L_\alpha < \tilde{b}^L_\alpha \\ \tilde{a}^U_\alpha < \tilde{b}^U_\alpha\end{cases} \text{or} \begin{cases}\tilde{a}^L_\alpha \leq \tilde{b}^L_\alpha \\ \tilde{a}^U_\alpha < \tilde{b}^U_\alpha\end{cases} \text{or} \begin{cases}\tilde{a}^L_\alpha < \tilde{b}^L_\alpha \\ \tilde{a}^U_\alpha \leq \tilde{b}^U_\alpha\end{cases}.$$

Definition 2.6 (Saito and Ishii, 2001) Let $\tilde{a} \in F(R)$ be a triangular fuzzy number then ithas membership function as below:

Research Article

$$\mu_{\tilde{a}}(r) = \begin{cases} \dfrac{r-a}{b-a}, & if \;\; a \le r \le b, \\ \dfrac{c-r}{c-b}, & if \;\; b < r \le c \end{cases}$$

which is denoted by $\tilde{a} = (a, b, c)$ and the α-level set of $\tilde{a}$ is:
$\tilde{a}_\alpha = [(1-\alpha)a + \alpha b, (1-\alpha)c + \alpha b]$.

Definition 2.7 (Pirzada and Pathak, 2013) Suppose that *V* be a real vector space and *F(R)* be the set of all fuzzy numbers.

Then a function $\tilde{f}: V \to F(R)$ is called fuzzy-valued function defined on *V*. Corresponding to such a function $\tilde{f}$ and $\alpha \in [0,1]$, we define two real-valued functions $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ on V as $\tilde{f}_\alpha^U(x) = (\tilde{f}(x))_\alpha^U$ and $\tilde{f}_\alpha^L(x) = (\tilde{f}(x))_\alpha^L$ for all $x \in V$.

Definition 2.8 (Diamond and Kloeden, 1994) Let $\tilde{f}: R^n \to F(R)$ be a fuzzy-valued function. We say that $\tilde{f}$ is continuous at $c \in R^n$ if for every $\varepsilon > 0$, there exists a $\delta = \delta(c, \varepsilon) > 0$ such that:
$d_F(\tilde{f}(x), \tilde{f}(c)) < \varepsilon$ for all $x \in R^n$ with $\|x - c\| < \delta$. That is,
$\lim_{x \to c} \tilde{f}(x) = \tilde{f}(c)$,
where $d_F(\tilde{a}, \tilde{b}) = sup_{0 \le \alpha \le 1} \, max\{|\tilde{a}_\alpha^L - \tilde{b}_\alpha^L|, |\tilde{a}_\alpha^U - \tilde{b}_\alpha^U|\}$ for all $\tilde{a}, \tilde{b} \in F(R)$, is the metric on $F(R)$.

Proposition 2.9 Let $\tilde{f}: R^n \to F(R)$ be a fuzzy-valued function. If $\tilde{f}$ is continuous at $c \in R^n$ then $\tilde{f}_\alpha^L(x)$ and $\tilde{f}_\alpha^U(x)$ are continuous at c for all $\alpha \in [0,1]$.

Definition 2.10 Let *X* be a subset of *R*. A fuzzy-valued function $\tilde{f}: X \to F(R)$ is said to be H-differentiable at $x_0 \in X$ if there exists a fuzzy number $D\tilde{f}(x_0)$ such that the limits (with respect to metric d_F)

$$\lim_{h \to 0^+} \frac{1}{h} \odot [\tilde{f}(x_0 + h) \ominus_H \tilde{f}(x_0)] \; and \; \lim_{h \to 0^+} \frac{1}{h} \odot [\tilde{f}(x_0) \ominus_H \tilde{f}(x_0 - h)]$$

both exist and are equal to $D\tilde{f}(x_0)$. In this case, $D\tilde{f}(x_0)$ is called the H-derivative of $\tilde{f}$ at x_0. If $\tilde{f}$ is H-differentiable at any $x \in X$, we call $\tilde{f}$ is H-differentiable over *X*.

Proposition 2.11 Let X be a subset of R. If a fuzzy-valued function $\tilde{f}: X \to F(R)$ is H-differentiable at $x_0 \in X$ with H-derivative $D\tilde{f}(x_0)$, then $\tilde{f}_\alpha^L(x)$ and $\tilde{f}_\alpha^U(x)$ are differentiable at x_0, for all $\alpha \in [0,1]$. Moreover, we have

$$(D\tilde{f})_\alpha(x_0) = [D(\tilde{f}_\alpha^L)(x_0), D(\tilde{f}_\alpha^U)(x_0)].$$

Definition 2.12 (Zimmermann, 1978) Let $\tilde{f}$ be a fuzzy-valued function defined on an open subset *X* of R^n and let $x_0 = (x_1^0, x_2^0, \ldots, x_n^0) \in X$ be fixed. We say that $\tilde{f}$ has the i^{th} partial H-derivative at x_0 if the fuzzy-valued function $\tilde{g}(x_i) = \tilde{f}(x_1^0, \ldots, x_{i-1}^0, x_i, x_{i+1}^0, \ldots, x_n^0)$ is H-differentiable at x_0 with H-derivative $D_i\tilde{f}(x_0)$. We also write $D_i\tilde{f}(x_0)$ as $(\frac{\partial \tilde{f}}{\partial x_i})(x_0)$.

Definition 2.13 We say that $\tilde{f}$ is H-differentiable at $x_0 \in X \subseteq R^n$ if one of the partial H-derivatives $\frac{\partial \tilde{f}}{\partial x_i}; i = 1, \ldots, n$, exists at x_0 and the remaining n-1 partial H-derivatives exist on some neighborhoods of x_0 and are continuous at x_0. The gradient of $\tilde{f}$ at x_0 is denoted by

$$\nabla \tilde{f}(x_0) = \left(D_1\tilde{f}(x_0), \ldots, D_n\tilde{f}(x_0)\right).$$

The α-level set of $\nabla \tilde{f}(x_0)$ is defined and denoted by

$(\nabla \tilde{f}(x_0))_\alpha = \left(\left(D_1\tilde{f}(x_0)\right)_\alpha, \ldots, \left(D_n\tilde{f}(x_0)\right)_\alpha\right)$, for all $\alpha \in [0,1]$.

where $(D_i\tilde{f}(x_0))_\alpha = [D_i\tilde{f}_\alpha^L(x_0), D_i\tilde{f}_\alpha^U(x_0)]$, $i = 1, \ldots, n$.

Proposition 2.14 Let $X \in R^n$ be an open subset. If fuzzy-valued function $\tilde{f}: X \to F(R)$ is H-differentiable on X then $\tilde{f}_\alpha^L(x)$ and $\tilde{f}_\alpha^U(x)$ are differentiable on X for all $\alpha \in [0,1]$. Moreover for each $\bar{x} \in X$, $(D_i\tilde{f}(\bar{x}))_\alpha = [D_i\tilde{f}_\alpha^L(\bar{x}), D_i\tilde{f}_\alpha^U(\bar{x})]$, $i = 1, \ldots, n$.

Definition 2.15 We say that $\tilde{f}$ is continuously H-differentiable at $\bar{x} \in X \subseteq R^n$if all ofthe partial H-derivatives$\left(\frac{\partial \tilde{f}}{\partial x_i}\right)(\bar{x}), i = 1, \dots, n$, exist on some neighborhoods of$\bar{x}$ and are continuous at $\bar{x}$. We say that$\tilde{f}$ is continuously H-differentiable on X if it is continuously H-differentiable at every$x \in X$.
Proposition 2.16 If $\tilde{f}$ is continuously H-differentiable on *X*, then$\tilde{f}_\alpha^L$and $\tilde{f}_\alpha^U$are also continuously differentiable on *X*, for all $\alpha \in [0,1]$.
Fuzzy Nonlinear Optimization
Let T be an open subset of $R^n$and$\tilde{f}$ be a fuzzy-valued function on *T*. Consider the following nonlinear equality-constrained fuzzy optimization problem:

$\min \tilde{f}(x) = \tilde{f}(x_1, x_2, \dots, x_n),$
$S.t.$
$\tilde{c}_i(x) = \tilde{0}, i \in E.$ (1)

Where$\tilde{0}$ defined as $\tilde{0}(r) = 1$ if $r = 0$ and $\tilde{0}(r) = 0$ if $r \neq 0$ and itslevel set is$\tilde{0}_\alpha = \{0\}$ for all $\alpha \in [0,1]$.
Definition 3.1 Let$x_0 \in \{x \in T | \tilde{c}_i(x) = \tilde{0}, i \in E\}$. We say that x_0 is a non-dominated solution of the problem (1) if there exists no $x_1 (\neq x_0) \in T$ such that $\tilde{f}(x_1) \prec \tilde{f}(x_0)$.
Definition 3.2 Let T be a convex subset of R^n and $\tilde{f}$ be a fuzzy-valued function defined on T. We say that $\tilde{f}$ is convex at x_0 if:
$\tilde{f}(\lambda x_0 + (1-\lambda)x) \preccurlyeq \lambda \odot \tilde{f}(x_0) \oplus (1-\lambda) \odot \tilde{f}(x)$:for each $\lambda \in (0,1)$ and $x \in T$.
Proposition 3.3 $\tilde{f}: T \subseteq R^n \to F(R)$ is convex at $x_0 \in T$ if and only if $\tilde{f}_\alpha^L(x)$ and $\tilde{f}_\alpha^U(x)$ are convex at x_0, for all $\alpha \in [0,1]$.
Theorem 3.4 Let the fuzzy-valued objective function $\tilde{f}: T \to F(R)$ be convex and continuously H-differentiable, where $T \subseteq R^n$ is open and convex.The fuzzy-valued constraints functions $\tilde{c}_i: T \to F(R) (i \in E)$ are convex and continuously H-differentiable. Let $X = \{x \in T \subset R^n | \tilde{c}_i(x) = \tilde{0}, i \in E\}$ be a feasible set of problem (1) and let $x_0 \in X$. Suppose there is some $x \in T$ such that $\tilde{c}_i(x) = \tilde{0}, i \in E$. Then x_0 is a non-dominated solution of problem (1) over X if and only if there exist multipliers $\lambda_i \in R, i \in E$, such that the Karush-Kuhn-Tucker first order conditions hold:

(i) $\int_0^1 (\nabla \tilde{f}_\alpha^L(x_0) + \nabla \tilde{f}_\alpha^U(x_0)) d\alpha + \sum_{i \in E} \lambda_i (\nabla \tilde{c}_{i\alpha}^L(x_0) + \nabla \tilde{c}_{i\alpha}^U(x_0)) = 0,$
(ii) $\tilde{c}_{i\alpha}^L(x_0) = 0, \tilde{c}_{i\alpha}^U(x_0) = 0;\ i \in E$, for all $\alpha \in [0,1]$.

Proof
Necessary Weset:

$$F(x) = \int_0^1 (\tilde{f}_\alpha^L(x) + \tilde{f}_\alpha^U(x)) d\alpha. \quad (2)$$

Since $\tilde{f}$ is convex and continuously H-differentiable function, by propositions 2.16 and 3.3, we say that *F(x)* is convex and continuously differentiable real-valued function on *T*. Since x_0 is a non-dominated solution of(1). Then there exists no $x_1 (\neq x_0) \in T$such that for all$\alpha \in [0,1]$:

$$\begin{cases} \tilde{f}_\alpha^L(x_1) < \tilde{f}_\alpha^L(x_0) \\ \tilde{f}_\alpha^U(x_1) \leq \tilde{f}_\alpha^U(x_0) \end{cases} \quad or \quad \begin{cases} \tilde{f}_\alpha^L(x_1) \leq \tilde{f}_\alpha^L(x_0) \\ \tilde{f}_\alpha^U(x_1) < \tilde{f}_\alpha^U(x_0) \end{cases} \quad or \quad \begin{cases} \tilde{f}_\alpha^L(x_1) < \tilde{f}_\alpha^L(x_0) \\ \tilde{f}_\alpha^U(x_1) < \tilde{f}_\alpha^U(x_0) \end{cases}.$$

That is, there is no $x_1 (\neq x_0) \in T$ such that:

$F(x_1) < F(x_0)$ (3)

Therefore:$F(x_0) \leq F(x_1)$.
Since $\tilde{c}_i$ are convex and continuously H-differentiable functions for$i \in E$, so it implies that $\tilde{c}_{i\alpha}^L\ and\ \tilde{c}_{i\alpha}^U$ are real-valued convex and continuously differentiable functions for $\alpha \in [0,1]$and$i \in E$,on the other hand, we have:

$$X = \left\{x \in T \subset R^n \middle| \tilde{c}_i(x) = \tilde{0}, i \in E\right\}$$
$$= \left\{x \in T \subset R^n \middle| \tilde{c}_{i\alpha}^L(x) = 0\ and\ \tilde{c}_{i\alpha}^U(x) = 0, i \in E\right\}.$$

Therefore, our problem becomes an optimization problem with real objective function *F(x)* subject to real constraints,$\tilde{c}_{i\alpha}^L(x) = 0\ and\ \tilde{c}_{i\alpha}^U(x) = 0, i \in E$. So, by Theorem 12.5 (Pathak and Pirzada, 2011), (KKT

Indian Journal of Fundamental and Applied Life Sciences ISSN: 2231– 6345 (Online)
An Open Access, Online International Journal Available at www.cibtech.org/sp.ed/jls/2015/04/jls.htm
2015 Vol. 5 (S4), pp. 1256-1265/Akrami and Hosseini
Research Article

conditions for real optimization problems) there exist multipliers $0 \leq \lambda_i \in R, i \in E$, such that the following Karush-Kuhn-Tucker first order conditions hold:
(i) $\nabla F(x_0) + \sum_{i\in E} \lambda_i(\nabla \tilde{c}^L_{i\alpha}(x_0) + \nabla \tilde{c}^U_{i\alpha}(x_0)) = 0$,
(ii) $\tilde{c}^L_{i\alpha}(x_0) = 0, \tilde{c}^U_{i\alpha}(x_0) = 0;\ i \in E$, for all $\alpha \in [0,1]$.
Sufficient We can prove this part by contradiction. Let $x_0$is not a non-dominated solution. Then there exists a $x_1(\neq x_0) \in T$ such that $\tilde{f}(x_1) \prec \tilde{f}(x_0)$. Therefore, for all $\alpha \in [0,1]$ we have

$$\tilde{f}^L_\alpha(x_1) + \tilde{f}^U_\alpha(x_1) < \tilde{f}^L_\alpha(x_0) + \tilde{f}^U_\alpha(x_0)$$

We obtain,
$F(x_1) < F(x_0)$
Since F is convex and continuously differentiable function. In addition,
$x_0 \in X = \{x \in T \subset R^n \big| \tilde{c}_i(x) = \tilde{0}, i \in E\}$.
By conditions (i) and (ii) of this theorem, we obtain the following new conditions:
a. $\nabla F(x_0) + \sum_{i\in E} \lambda_i(\nabla \tilde{c}^L_{i\alpha}(x_0) + \nabla \tilde{c}^U_{i\alpha}(x_0)) = 0$,
b. $\tilde{c}^L_{i\alpha}(x_0) = 0, \tilde{c}^U_{i\alpha}(x_0) = 0;\ i \in E$, for all $\alpha \in [0,1]$.
Using theorem 12.5 Pathak and Pirzada, (2011), we say that x_0 is an optimal solution of real-objective function F with real constraints $\tilde{c}^L_{i\alpha}(x_0) = 0, \tilde{c}^U_{i\alpha}(x_0) = 0;\ i \in E$, for all $\alpha \in [0,1]$, i.e. $F(x_0) \leq F(x_1)$, which contradicts to (3). Hence, the proof is completed.☐
Let Tbe an open subset of $R^n$and$\tilde{f}$ be a fuzzy-valued function on T. Consider the following nonlinear unconstrained fuzzy optimization problem:
$\min \tilde{f}(X) = \tilde{f}(x_1, x_2, \dots, x_n)$,
$S.t.\ \ X \in T$. (4)
For solving the problem (4), we will use of the Newton method that described by Pirzade and Pathak, (2013). A locally non-dominated solution of the problem (4) is given as follows.

Definition 3.5 Let Tbe an open subset ofR^n.
(i) A point $\bar{x} \in T$ is a locally non-dominated solution of (4) if there exists no $x^0(\neq \bar{x}) \in N_\epsilon(\bar{x}) \cap T$ such that: $\tilde{f}(x^0) \prec \tilde{f}(\bar{x})$. Where $N_\epsilon(\bar{x})$ is a ϵ-neighborhood of $\bar{x}$.
(ii) A point $\bar{x} \in T$ is a non-dominated solution of (4) if there exists no $x^0(\neq \bar{x}) \in T$such that:$\tilde{f}(x^0) \prec \tilde{f}(\bar{x})$.

(iii) A point $\bar{x} \in T$ is a locally weak non-dominated solution of (4) if there exist no $x^0(\neq \bar{x}) \in N_\epsilon(\bar{x}) \cap T$such that: $\tilde{f}(x^0) \preccurlyeq \tilde{f}(\bar{x})$.

(iv) A point $\bar{x} \in T$ is a weak non-dominated solution of (4) if there exists no$x^0(\neq \bar{x}) \in T$such that: $\tilde{f}(x^0) \preccurlyeq \tilde{f}(\bar{x})$.

Quadratic Fuzzy Penalty Method
We describe the quadratic fuzzy penalty method first in the context of the equality-constrained problem:
$\min_x \tilde{f}(x)$,
$S.t.$
$\tilde{c}_i(x) = \tilde{0}, i \in E.$ (5)
Where $\tilde{f}(x)$ is a fuzzy function and , $\tilde{c}_i(x) = \tilde{0}, i \in E$, are fuzzy constraints.
The quadratic fuzzy penalty function for this formulation is:
$\tilde{Q}(x,\mu) = \tilde{f}(x) + \frac{\mu}{2}\sum_{i\in E} \tilde{c}^2_i\,(x)$. (6)
Where $\mu > 0$ is penalty parameter. By driving $\mu$to ∞, we penalize the constraint violations with increasing severity. It makes good intuitive sense to consider a sequence of values $\{\mu_k\}$ with $\mu_k \uparrow \infty$ as $k \to \infty$, and to seek the approximate minimizer x_k of $\tilde{Q}(x,\mu_k)$ for each k. Because the penalty terms in (6) are smooth, we can use techniques from unconstrained fuzzy optimization to search for x_k. For suitable choices of the

Indian Journal of Fundamental and Applied Life Sciences ISSN: 2231– 6345 (Online)
An Open Access, Online International Journal Available at www.cibtech.org/sp.ed/jls/2015/04/jls.htm
2015 Vol. 5 (S4), pp. 1256-1265/Akrami and Hosseini
Research Article

sequence $\{\mu_k\}$ and the initial guesses, just a few steps of unconstrained fuzzy minimization may be needed for each $\{\mu_k\}$.
For the general constrained nonlinear fuzzy optimization problem:
$\min_x \tilde{f}(x),$
$S.t.$
$\tilde{c}_i(x) = \tilde{0}, i \in E,$
$\tilde{c}_i(x) \geqslant \tilde{0}, i \in I.$ (7)
We can define the quadratic fuzzy penalty function for the problem (7) as:
$\tilde{Q}(x;\mu) = \tilde{f}(x) + \frac{\mu}{2}\sum_{i\in E}\tilde{c}_i^2\,(x) + \frac{\mu}{2}\sum_{i\in I}([\tilde{c}_i(x)]^-)^2$ (8)
where $[\tilde{y}]^- = \max\{-\tilde{y}, \tilde{0}\}$.
In this case, $\tilde{Q}$ may be less smooth than the objective and constraint functions.
*Algorithmic Framework:*A general framework for algorithms based on the quadratic fuzzy penalty function (6) can be specified as follows.
***Framework 4.1.1** (Quadratic fuzzy penalty method)*
Given $\mu_0 > 0$, nonnegative sequences$\{\tau_{k_1}\}, \{\tau_{k_2}\}$ with$\tau_{k_1}, \tau_{k_2} \to 0$, and a starting point x_0^s.
For$k = 0,1,2, \ldots$
Find an approximate minimizer x_k of $\tilde{Q}(x, \mu_k)$, starting at x_k^s.
and terminating when:
$\|\nabla_x \tilde{Q}_\alpha^U(x,\mu_k)\| \le \tau_k \; and \; \|\nabla_x \tilde{Q}_\alpha^L(x,\mu_k)\| \le \tau_k$ for all $\alpha \in [0,1]$; where: $\tau_k = \min\{\tau_{k_1}, \tau_{k_2}\}$.
If final convergence test satisfied
Stop with approximate solution x_k;
End (If)
Choose new penalty parameter $\mu_{k+1} > \mu_k$;
Choose new starting point x_{k+1}^s
End (For)
The parameter sequence $\{\mu_k\}$can be chosen adaptively, based on the difficulty ofminimizing the penalty function at each iteration. When minimization of $\tilde{Q}(x, \mu_k)$provesto be expensive for some k, we choose μ_{k+1} to be only modestly larger thanμ_k; for instance $\mu_{k+1} = 1.5\mu_k$. If we find the approximate minimizer of $\tilde{Q}(x, \mu_k)$ cheaply, we could try a more ambitious increase, for instance$\mu_{k+1} = 10\mu_k$. The convergence theory for Framework 4.1.1 allows wide latitude in the choice of nonnegative tolerances $\tau_k = \min\{\tau_{k_1}, \tau_{k_2}\}$;it requires only that $\tau_k \to 0$, to ensure that the minimization is carried out more accurately as the iterations progress.
When only equality constraints are present, $\tilde{Q}(x, \mu_k)$ is smooth, so the algorithms for unconstrained fuzzy minimization described by (Pathak and Pirzada, 2013)can be used to identify the approximate solutionx_k. However, the minimization of $\tilde{Q}(x, \mu_k)$ becomes more difficult to perform as μ_k becomes large, unless we use special techniques to calculate the search directions. For one thing, theHessian $\nabla_{xx}^2\tilde{Q}(x, \mu_k)$becomes arbitrarily ill conditioned near the minimizer. This property alone is enough to make unconstrained fuzzy minimization algorithms such as Newton's method proposed by (Pathak and Pirzada, 2013).This method, is not sensitive to ill conditioning of the Hessian, but it, too, may encounter difficulties for large μ_k.
Convergence of Tthe Proposed Method
We describe some convergence properties of the quadratic fuzzy penalty method in the following two theorems. We restrict our attention to the equality-constrained problem (5), for which the quadratic fuzzy penalty function is defined by (6). For the first result, we assume that the fuzzy penalty function $\tilde{Q}(x, \mu_k)$ has a finite minimizer for each value of μ_k.
Theorem 5.1 Suppose that each x_k is the exact global minimizer of $\tilde{Q}(x, \mu_k)$ defined by (6) in framework 4.1.1, and $\mu_k \uparrow \infty$. Then every limit point x^* of the sequence $\{x_k\}$ is a global solution of the problem (5).
Proof Let $\bar{x}$ be a global solution of (5), that is,

$\tilde{f}(\bar{x}) \preccurlyeq \tilde{f}(x)$, for all x with $\tilde{c}_i(x) = \tilde{0}, i \in E$.
Since x_k minimizes $\tilde{Q}(.;\mu_k)$ for each k, we have $\tilde{Q}(x_k;\mu_k) \preccurlyeq \tilde{Q}(\bar{x};\mu_k)$, which leads to the inequality
$\tilde{f}(x_k) + \frac{\mu_k}{2}\sum_{i\in E} \tilde{c}_i^2(x_k) \preccurlyeq \tilde{f}(\bar{x}) + \frac{\mu_k}{2}\sum_{i\in E} \tilde{c}_i^2(\bar{x}) = \tilde{f}(\bar{x})$ (9)
By rearranging this expression, we obtain
$\sum_{i\in E} \tilde{c}_i^2(x_k) \preccurlyeq \frac{2}{\mu_k}[\tilde{f}(\bar{x}) - \tilde{f}(x_k)]$. (10)
Suppose that x^* is a limit point of $\{x_k\}$, so there is an infinite subsequence K such that
$\lim_{k\in K} x_k = x^*$.
By taking the limit as $k \to \infty$, $k \in$ K, on both sides of (10), we obtain

$$\sum_{i\in E} \tilde{c}_i^2(x^*) = \lim_{k\in K}\sum_{i\in E} \tilde{c}_i^2(x_k) \preccurlyeq \lim_{k\in K}\frac{2}{\mu_k}[\tilde{f}(\bar{x}) - \tilde{f}(x_k)] = 0,$$

where the last equality follows from $\mu_k \uparrow \infty$. Therefore, we have $\tilde{c}_i(x^*) = \tilde{0}$, for all $i \in E$, so x^* is a feasible point. Moreover, by taking the limit as $k \to \infty$, for k ∈ K in (9), we have by no negativity of μ_k and of each $\tilde{c}_i^2(x_k)$ that

$$\tilde{f}(x^*) \preccurlyeq \tilde{f}(x^*) + \lim_{k\in K}\frac{\mu_k}{2}\sum_{i\in E} \tilde{c}_i^2(x_k) \preccurlyeq \tilde{f}(\bar{x}).$$

Since x^* is a feasible point whose objective value is no larger than that of the global solution $\bar{x}$, we conclude that x^*, too, is a global solution, as claimed.☐

Since this result requires us to find the global minimizer for each sub-problem, this desirable property of convergence to the global solution of (5) cannot be attained in general. The next result concerns convergence properties of the sequence $\{x_k\}$ when we allow inexact (but increasingly accurate) minimizations of $\tilde{Q}(.;\mu_k)$. In contrast to Theorem 5.1, it shows that the sequence may be attracted to infeasible points, or to any KKT point (that is, a point satisfying first-order necessary conditions of theorem 3.4) rather than to a minimizer. It also shows that the quantities $\mu_k\tilde{c}_i(x_k)$ may be used as estimates of the Lagrange multipliers$\tilde{\lambda}_i^*$ in certain circumstances. To establish the result we will make the (optimistic) assumption that the stop test in framework 4.1.1 is satisfied.

Definition 5.2 Let $\tilde{f}: T \subseteq R \to F(R)$ be a fuzzy-valued function. Then we say $x^* \in T$ is a stationary point of the function $\tilde{f}$. If $0 \in (\tilde{f}(x^*)')_\alpha$ for some value of $\alpha \in [0,1]$. On the other hand $x^* \in T$ is a stationary point of the function $\tilde{f}$ if $\tilde{f}'(x^*) = \tilde{0}$.

Theorem 5.3 Suppose that the tolerances and penalty parameters in Framework 4.1.1 satisfy $\min\{\tau_{k_1}, \tau_{k_2}\} = \tau_k \to 0$ and $\mu_k \uparrow \infty$. Then if a limit point x^* of the sequence $\{x_k\}$ is infeasible, it is a stationary point of the function $\|\tilde{c}(x)\|^2$. On the other hand, if a limit point x^* is feasible and the constraint gradients $\nabla\tilde{c}_i(x^*)$ are linearly independent, then x^* is a KKT point for the problem (5). For such points, we have for any infinite subsequence K such that $\lim_{k\in K} x_k = x^*$ that
$\lim_{k\in K} \mu_k\tilde{c}_i(x_k) = -\tilde{\lambda}_i^*$ for all $i \in E$. (11)
Where $\tilde{\lambda}^*$ is the multiplier vector that satisfies the KKT conditions**,** for the problem (5).

Proof By differentiating $\tilde{Q}(x;\mu_k)$ in (6), we obtain
$\nabla_x\tilde{Q}(x_k,\mu_k) = \nabla\tilde{f}(x_k) + \sum_{i\in E}\mu_k\tilde{c}_i(x_k)\nabla\tilde{c}_i(x_k)$ (12)
Therefore, from the termination criterion for Framework 4.1.1, we have
$\left\|\nabla\tilde{f}_\alpha^U(x_k) + \sum_{i\in E}\mu_k\tilde{c}_{i\alpha}^U(x_k)\nabla\tilde{c}_{i\alpha}^U(x_k)\right\| \le \tau_k$, for all $\alpha \in [0,1]$. (13)
where $\tau_k = \min\{\tau_{k_1}, \tau_{k_2}\}$.
By rearranging this expression (and in particular using the inequality $\|a\| - \|b\| \le \|a+b\|$) we obtain
$\left\|\sum_{i\in E}\mu_k\tilde{c}_{i\alpha}^U(x_k)\nabla\tilde{c}_{i\alpha}^U(x_k)\right\| \le \frac{1}{\mu_k}\left[\tau_k + \left\|\nabla\tilde{f}_\alpha^U(x_k)\right\|\right]$, for all$\alpha \in [0,1]$. (14)
Let x^* be a limit point of the sequence of iterates. Then there is a subsequence K such that $\lim_{k\in K} x_k = x^*$.
When we take limits as $k \to \infty$, for k ∈ K, the bracketed term on the right-hand-side approaches

Indian Journal of Fundamental and Applied Life Sciences ISSN: 2231– 6345 (Online)
An Open Access, Online International Journal Available at www.cibtech.org/sp.ed/jls/2015/04/jls.htm
2015 Vol. 5 (S4), pp. 1256-1265/Akrami and Hosseini
Research Article

to$\left\|\nabla\tilde{f}_{\alpha}^{U}(x^{*})\right\|$, so because $\mu_k \uparrow \infty$, the right-hand-side approaches to zero. From the corresponding limit on the left-hand-side, we obtain
$\sum_{i\in E}\mu_k\tilde{c}_{i\alpha}^{U}(x^{*})\nabla\tilde{c}_{i\alpha}^{U}(x^{*}) = 0$,for all $\alpha \in [0,1]$.(15)
Similarly, we have:
$\sum_{i\in E}\mu_k\tilde{c}_{i\alpha}^{L}(x^{*})\nabla\tilde{c}_{i\alpha}^{L}(x^{*}) = 0$,for all $\alpha \in [0,1]$. (16)
Therefore, $\tilde{c}_i(x^{*}) \neq \tilde{0}$ (if the constraints gradients $\nabla\tilde{c}_i(x^{*})$ are dependent), but in this case, (15) and (16) implies that x^{*} is a stationary point of the function $\|\tilde{c}(x)\|^2$.
On the other hand, if the constraint gradients $\nabla\tilde{c}_i(x^{*})$ are linearly independent at a limit point x^{*}, we have from (15) and (16) that $\tilde{c}_i(x^{*}) = \tilde{0}$ for all $i \in E$, so x^{*} is a feasible point. Hence, the second KKT condition (ii) of theorem 3.4 is satisfied. We need to check the first KKT condition (i) of theorem 3.4 as well, and to show that the limit (11) holds. By using $\tilde{A}(x)$ to denote the matrix of constraint gradients (known as the Jacobin), that is,
$\tilde{A}(x)^T = [\nabla\tilde{c}_i(x)]_{i\in E}$ (17)
and $\tilde{\lambda}_k$ to denote the vector $\mu_k\tilde{c}(x_k)$, we have as in (5.5), for all $\alpha \in [0,1]$:
$-\tilde{A}_{\alpha}^{U}(x_k)^T\tilde{\lambda}_{k\alpha}^{U} = \nabla\tilde{f}_{\alpha}^{U}(x_k) - \nabla_x\tilde{Q}_{\alpha}^{U}(x_k,\mu_k)$; $\left\|\nabla_x\tilde{Q}_{\alpha}^{U}(x,\mu_k)\right\| \le \tau_k$ (18)
For all $k \in K$ sufficiently large, the matrix $\tilde{A}_{\alpha}^{U}(x_k)$ has full row rank, so $\tilde{A}_{\alpha}^{U}(x_k)\tilde{A}_{\alpha}^{U}(x_k)^T$ is nonsingular, for all $\alpha \in [0,1]$.
By multiplying $\tilde{A}_{\alpha}^{U}(x_k)$ to both sides of (18) and rearranging, we have
$\tilde{\lambda}_{k\alpha}^{U} = -\left[\tilde{A}_{\alpha}^{U}(x_k)\tilde{A}_{\alpha}^{U}(x_k)^T\right]^{-1}[\tilde{A}_{\alpha}^{U}(x_k)\nabla\tilde{f}_{\alpha}^{U}(x_k) + \nabla_x\tilde{Q}_{\alpha}^{U}(x_k,\mu_k)]$.
Hence, by taking the limit as $k \in K$ goes to ∞, we find
$\tilde{\lambda}_{\alpha}^{*U} = \lim_{k\in K}\tilde{\lambda}_{k\alpha}^{U} = -[\tilde{A}_{\alpha}^{U}(x^{*})\tilde{A}_{\alpha}^{U}(x^{*})^T]^{-1}\tilde{A}_{\alpha}^{U}(x^{*})\nabla\tilde{f}_{\alpha}^{U}(x^{*})$.
Similarly, we can show that
$\tilde{\lambda}_{\alpha}^{*L} = \lim_{k\in K}\tilde{\lambda}_{k\alpha}^{L} = -[\tilde{A}_{\alpha}^{L}(x^{*})\tilde{A}_{\alpha}^{L}(x^{*})^T]^{-1}\tilde{A}_{\alpha}^{L}(x^{*})\nabla\tilde{f}_{\alpha}^{L}(x^{*})$.
By taking limits in (13), we conclude that
$\nabla\tilde{f}_{\alpha}^{U}(x^{*}) + \tilde{A}_{\alpha}^{U}(x^{*})^T\tilde{\lambda}_{\alpha}^{*U} = 0$, for all $\alpha \in [0,1]$.
Similarly, we have
$\nabla\tilde{f}_{\alpha}^{L}(x^{*}) + \tilde{A}_{\alpha}^{L}(x^{*})^T\tilde{\lambda}_{\alpha}^{*L} = 0$, for all $\alpha \in [0,1]$. (19)
So that $\tilde{\lambda}^{*}$ satisfies the first KKT condition (i) of theorem 3.4 for (5). Hence, x^{*} is a KKT point for (5), with unique Lagrange multiplier vector $\tilde{\lambda}^{*}$☐.

Numerical Examples

In this section, we present some examples and use proposed method in section 4 to solve them. In addition, the Newton method for solving the unconstrained fuzzy optimization problems (Pathak and Pirzada, 2013) is used.
Example 1 Consider the following fuzzy optimization problem with one equality constraints:
$\min\ \tilde{1}\odot x_1 \oplus \tilde{1}\odot x_2$,
$S.t.$
$\tilde{1}\odot x_1^2 \oplus \tilde{1}\odot x_2^2 \ominus_H \tilde{2} = \tilde{0}$,
$x_1, x_2 \ge 0$.(20)
Where $\tilde{1} = (0,1,2)$ and $\tilde{2} = (0,2,4)$ are triangular fuzzy numbers.
Here, the quadratic fuzzy penalty function is:
$\tilde{Q}(x;\mu) = \tilde{1}\odot x_1 \oplus \tilde{1}\odot x_2 \oplus \frac{\mu}{2}\left(\tilde{1}\odot x_1^2 \oplus \tilde{1}\odot x_2^2 \ominus_H \tilde{2}\right)^2$. (21)
By solving the problem (21) with Newton method, the optimal solution is obtained as $x^{*} = (1,1)^T$.
Example 2 Consider the following fuzzy optimization problem with two equality constraints:
$\min -\tilde{1}\odot x_1$
$S.t.$

$\tilde{1}\odot x_1^2 \oplus \tilde{1}\odot x_2^2 = \tilde{1}$,
$(\tilde{1}\odot x_1 \ominus_H \tilde{1})^3 \ominus_H \tilde{1}\odot x_2^2 = \tilde{0}$,
$x_1, x_2 \geq 0$.(22)
where$\tilde{1} = (0,1,2)$.
The quadratic fuzzy penalty function is:
$\tilde{Q}(x;\mu) = -\tilde{1}\odot x_1 \oplus \frac{\mu}{2}\odot[(\tilde{1}\odot x_1^2 \oplus \tilde{1}\odot x_2^2 \ominus_H \tilde{1})^2 \oplus ((\tilde{1}\odot x_1 \ominus_H \tilde{1})^3 \ominus_H \tilde{1}\odot x_2^2)^2] = \tilde{0}$. (23)
By solving the problem (23) with Newton method the optimal solution is obtained as $x^* = (1,0)^T$.
Example 3 Consider the following fuzzy optimization problem with equality and inequality constraints:
$\min -\tilde{2}\odot x_2$
$S.t.$
$\tilde{1}\oplus\tilde{1}\odot x_1 \ominus_H \tilde{2}\odot x_2 \succcurlyeq \tilde{0}$,
$\tilde{1}\odot x_1^2 \oplus \tilde{1}\odot x_2^2 \ominus_H \tilde{1} = \tilde{0}$,
$x_1, x_2 \geq 0$.(24)
Where $\tilde{1} = (0,1,2)$ and $\tilde{2} = (0,2,4)$ are triangular fuzzy numbers.
The quadratic fuzzy penalty function is:
$\tilde{Q}(x;\mu) = (-\tilde{2}\odot x_2) \oplus \frac{\mu}{2}\odot[(\tilde{1}\odot x_1^2 \oplus \tilde{1}\odot x_2^2 \ominus_H \tilde{1})^2 \oplus ([\tilde{1}\oplus\tilde{1}\odot x_1 \ominus_H \tilde{2}\odot x_2]^-)^2 = \tilde{0}$. (25)
By solving the problem (25) with Newton method the optimal solution is obtained as $x^* = (0.6,0.8)^T$.

CONCLUSION

In this paper, a quadratic fuzzy penalty method presented for solving nonlinear fuzzy programming problems. We have proved the convergence of the method and presented an algorithm for the same. Appropriate illustrations are given to justify the proposed method.

ACKNOWLEDGEMENT

This research was financially supported by the center of Excellence for Robust Intelligent systems of Yazd University. The authors wish to thank Prof. S. M. Karbassi for his constructive comments.

REFERENCES

Behera SK and Nayak JR (2012). Optimal solution of fuzzy non linear programming problems with linear constraints. *International Journal of Advances in Science and Technology* **4**(2) 43-91.

Charnes A and Cooper WW (1959). Chance-constrained programming. *Management Science*.[Online] **6**Available: http://dx.doi.org/10.1287/mnsc.6.1.73 73-79.

Cho Gyeong Mi (2005). Log-barrier method for two-stage quadratic stochastic programming. *Applied Mathematics and Computation* **164**(1)45–69.

Delgado M, Verdegay JL and Vila MA (1989). A general model for fuzzy linear programming. *Fuzzy Sets and Systems* **29**(1) 21–29.

Diamond P and Kloeden P (1999). Metric Spaces of Fuzzy Sets. *Fuzzy Sets and Systems* **100**(1) 63-71.

Jameel AF and Radhi AZ (2014). Penalty Function Method for Solving Fuzzy Nonlinear Programming Problem. *International Journal of Recent Research in Mathematics Computer Science and Information Technology* **28** 1-13.

Jamison KD and Lodwick WA (2001). Fuzzy linear programming using a penalty method. *Fuzzy Sets and Systems* **119**(1) 97-110.

Kall P (1982). Stochastic programming. *European Journal of Operational Research* **10** 125-130.

Kheirfam B (2011). A Method for solving fully fuzzy quadratic programming problems. *Actauniversitatis Apulensis* **27** 69-76.

Kumar A, Kaur J and Singh P (2010). Fuzzy Optimal Solution of Fully Fuzzy Linear Programming Problems with Inequality Constraints. *International Journal of Applied Mathematics and Computer Sciences***6**(1)37-40.

Research Article

Liu XW and Da QL (1999). A satisfactory solution for interval number linear programming. *Journal of Systems Engineering* **14** 123–128.
Liu BD and Iwamura K (2001). Fuzzy programming with fuzzy decisions and fuzzy simulation based genetic algorithm. *Fuzzy Sets and Systems* **122**(2) 253–262.
Luhandjula MK (1989). Fuzzy optimization: An appraisal. *Fuzzy Sets and Systems* **30**(3) 257-282.
Pathak VD and Pirzada UM (2011). Necessary and Sufficient Optimality Conditions for Non-linear Fuzzy Optimization Problem. *International Journal of Mathematical Science Education* **4**(1) 1 - 16.
Pirzada UM and Pathak VD (2013). Newton Method for Solving the Multi-Variable Fuzzy Optimization Problem. *Journal of Optimization Theory and Applications* **156** 867-881.
Saito S and Ishii H (2001). L-Fuzzy Optimization Problems by Parametric Representation. *IEEE* **2** 1173-1177.
Slowinski R (1986). A multicriteria fuzzy linear programming method for water supply systems development planning. *Fuzzy Sets and Systems* **19**(3) 217–237.
Stefanini L (2010). A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. *Fuzzy Sets and Systems* **161**(11) 1564 –1584.
Tanaka H, Okuda T and Asai K (1974). On Fuzzy Mathematical Programming. *The Journal of Cybernetic* **3** 37-46.
Zimmerman HJ (1978). Fuzzy programming and linear programming with several objective functions. *Fuzzy sets and systems* **1**(1) 45-55.