Indian Journal of Fundamental and Applied Life Sciences ISSN: 2231– 6345 (Online)
An Open Access, Online International Journal Available at www.cibtech.org/sp.ed/jls/2015/04/jls.htm
2015 Vol. 5 (S4), pp. 193-198/Gholamtabar and Parandin
Research Article

NUMERICAL SOLUTIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS BY PREDICTOR- CORRECTOR METHOD

Shaban Gholamtabar[1] and *Nouredin Parandin[2]
[1]Department of Mathematics, College of Science, Ayatollah Amoli Branch, Islamic Azad University Amol, Iran
[2]Department of Mathematics,College of Science, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran
**Author for Correspondence*

ABSTRACT
So far, many methods have been presented to solve the first-order differential equations. Not many studies have been conducted for numerical solution of high-order differential equations. In this research, first we have applied Adam Bashforth multi-step methods for the initial approximation of higher-order differential equations so to improve the approximation; we modify the method of Adam- moulton. To solve it, first we convert the equation to the first-order differential equation by order reduction method. Then we use a single-step method such as Euler, Taylor or Runge-Kutta for approximation of initially orders which are required to start Predictor- Corrector method. Now we can use the proposed method to approximate rest of the points. Finally, we examine the accuracy of method by presenting examples.

Keywords: *High-Order Differential Equations, Predictor- Corrector, Order Reduction*

INTRODUCTION
Differential equations are very useful indifferent sciences such as physics, chemistry, biology and economy. To learn more about the use of these equations in mentioned sciences, you can see the application of these equations in physics (Budd and Iserles, 1999; Kotikov, 1991; Gang and Kaifen, 1993; Its *et al*., 1990; Peng, 2003), chemistry in (Verwer *et al.*, 1996; Behlke and Ristau, 2002; Salzner *et al*., 1990), biology in (Culshaw and Ruan, 2000; Bocharov and Rihan, 2000) and economy in (Norberg, 1995). Considering that most of the time analytic solution of such equations and finding an exact solution has either high complexity or cannot be solved, we applied numerical methods for the solution. Due to our subject which is solving the second-order differential equations, we will refer to some solution methods which have been proposed in recent years by other researchers to solve the equations. In 1993, Zhang presented a solution method for second-order boundary value problems. In 2000, Yang introduced quasi-approximate periodic solutions for second-order neutral delay differential equations (Yuan, 2000). In 2003, Yang presented a method for solving second-order differential equations with almost periodic coefficients (Yuan, 2003). In 2005, Liu *et al.*, obtained periodic solutions for high-order delayed equations. In 2005, Nieto and Lopez used Green's function to solve second-order differential equations which boundary value is periodic. In 2006, Yang *et al.*, also applied Green's function to solve second-order differential equations. In 2008, Pan obtained periodic solutions for high-order differential equations with deviated argument. In 2011, Lopez used non-local boundary value problems for solving second-order functional differential equations. It should be noted that most of these equations have piecewise arguments. Other parts of this paper are organized as follows. In the second part, we will review the required definitions and basic concepts. In the third section, we will propose the basic idea for solve in second order differential equations. In the fourth section, we will provide examples for further explanation of the method and in fifth section; it will end with results of discussion.

Required Definitions and Basic Concepts
Definition 3.1 (Burden and Faires): The general form of a second-order differential equation is as follows:

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=R(t) \qquad (3\text{-}1)$$

Or it can be more simply stated as below:

$y'' + p(t)y'_0 + q(t)y = R(t)$

In which, p (t), q (t) and R (t) are functions of t and without any reduction in totality, the coefficient of y'' is equal to 1 because also in another way, this coefficient will already convert to one for y'' with dividing by coefficient of y''.

Note: I fine the equation (3.1) the value of R (t) is 0, then the equation is called a homogeneous equation.

Definition 3.2 (Burden and Faires): We say that the function of $f(t,y)$ with variable of y on series of $D \subset R^2$ is true in Lipshitz condition. If a fix such $L > 0$ exists with this property that when $(t,y_1),(t,y_2) \in D$,

$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2| \qquad (3\text{-}2)$$

We consider the fix L as a Lipschitz fix for f.

Definition 3.3 (Burden and Faires): we say that the set of $D \subset R^2$ is convex. If, whenever $(t_1,y_1),(t_2,y_2)$ belongs to D, the point of $((1-\lambda)t_1 + \lambda t_2 \,,\; (1-\lambda)y_1 + \lambda y_2)$ which $0 \leq \lambda \leq 1$ belongs to D per each λ.

Theorem 3.1 (Burden and Faires): Suppose that $f(t,y)$ is described on a convex set of $D \subset R^2$. If a fix such $L > 0$ exists that per each $(t,y) \in D$,

$$\left|\frac{\partial f}{\partial y}(t,y)\right| \leq L$$

Then f according to the variable of y on D in lipschitz condition is true with L fix lipschitz.

Theorem 3.2 (Burden and Faires): suppose that $D = \{(t,y) a \leq t \leq b, -\infty < y < \infty\}$ and $f(t,y)$ are contiguous on D. when f is true in lipschitz condition according to the variable of y on D, then problem of initial value of $y(a) = a$, $a \leq t \leq b$ and $y' = f(t,y)$ have unique solution of $y(t)$ per $a \leq t \leq b$.

Definition 3.4 (Burden and Faires): a multi-steps technique to solve the problem of initial value.

$$y' = f(t,y), \qquad a \leq t \leq b, \qquad y(a) = \alpha \qquad (3\text{-}3)$$

It is a technique that its' differential equation is to find the approximation of w_{i+1} in the network point of t_{i+1} which can be shown by below equation in which m is an integer greater

$$w_{i+1} = a_{n-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} + h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})] \qquad (3\text{-}4)$$

Per each $i = m-1, m, \ldots, N-1$ in which the initial values of

$$w_0 = \alpha_0, w_1 = \alpha_1, w_2 = \alpha_2, \ldots, w_{m-1} = \alpha_{m-1} \qquad (3\text{-}5)$$

Are determined and as typical $h = (b-a)/N$

When $b_m = 0$, we call it explicit or open method and in the equation (3-2) it gives the value of w_{i+1} explicitly based on predetermined values. When $b_m \neq 0$, we call it implicit or close method, because w_{i+1} appears in both sides of (3-4) and it can be determined only with an implicit method.

The Main Idea for Solving Second-Order Differential Equations

We consider the differential equations as following in which t is an independent variable and y is a dependant variable.

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt}) \qquad (4\text{-}1)$$

Now to solve such equations, we act as follows:

$$\frac{dy}{dt} = p \qquad (4.2)$$

Research Article

According to the equation (2-3), we convert the relation (1-3) to two first-order differential equation as follows:

$$\frac{dy}{dt} = p = f_1(t, y, p) \tag{4.3}$$

$$\frac{dp}{dt} = f_2(t, y, p) \tag{4.4}$$

Then we use Adam Bashforth's two-steps, three-steps and … method to solve the equations (4-3) and (4.4).

Suppose that the initial condition for (4-3) and (4-4) are given as below:

$$y(t_0) = y_0, y'(t_0) = p(t_0) = p_0 \tag{4.5}$$

Now, if we want to use n-step Adam Bashforth method for n= 2, 3, 4… in this case we should use n-1 initial step with a single-step method such as Euler, Taylor or Runge-Kutta. In this research, for example we describe the Runge-Kutta single-step method to approximate the n-1 initial step as following:

$$y(t_m) = y_m, y'(t_m) = p(t_m) = p_m \; 0 < m < n \tag{4.6}$$

Then we describe

$$\begin{cases} k_1 = hf_1(t_m, w_m, v_m) \\ . \\ k_2 = hf_1(t_m + h, w_m + k_1, v_m + l_1) \end{cases} \tag{4.7}$$

$$\begin{cases} k_2 = hf_2(t_m, w_m, v_m) \\ . \\ k_2 = hf_2(t_m + h, w_m + k_1, v_m + l_1) \end{cases} \tag{4.8}$$

Now suppose that we want to obtain the value of $v_{m+1}, w_{m+1}$by v_m, w_m, we have:

$$\begin{cases} w_{m+1} = w_m + \frac{1}{2}(k_1 + k_2) \\ . \\ v_{m+1} = v_m + \frac{1}{2}(l_1 + l_2) \end{cases} \tag{4.9}$$

Now suppose that we are in n step that after this step we want to use Adam Bashforth multi-step method as the driver, then we modify the method of Adam- moulton general form of these methods is as below:

$$\begin{cases} w_{i+1}^{(0)} = a_{n-1}w_i + a_{n-2}w_{i-1} + \cdots + a_0 w_{i-(n-1)} + h(b_{n-1}f_1(t_i, w_i, v_i) \\ \qquad + \cdots + b_0 f_1\left(t_{i-(n-1)}, w_{i-(n-1)}, v_{i-(n-1)}\right) \\ . \\ w_0 = \alpha_0, w_1 = \alpha_1, \dots, w_{n-1} = \alpha_{n-1} \end{cases} \tag{4.10}$$

$$\begin{cases} w_{i+1}^{(k)} = a_{n-1}w_i + a_{n-2}w_{i-1} + \cdots + a_0 w_{i-(n-1)} + h\left(b_n f_1(t_{i+1}, w_{i+1}^{(k-1)}, v_{i+1}^{(k-1)}\right) \\ \qquad + \cdots + b_0 f_1\left(t_{i-(n-1)}, w_{i-(n-1)}, v_{i-(n-1)}\right) \\ . \\ w_0 = \alpha_0, w_1 = \alpha_1, \dots, w_{n-1} = \alpha_{n-1} \end{cases} \tag{4.11}$$

That the relation (4.11) is determined per $i = n, n + 1, \dots, N$

And also

$$\begin{cases} v_{i+1}^{(0)} = c_{n-1}v_i + c_{n-2}v_{i-1} + \cdots + c_0 v_{i-(n-1)} + h(b_{n-1}f_2(t_i, w_i v_i) \\ \qquad + \cdots + b_0 f_2\left(t_{i-(n-1)}, w_{i-(n-1)}, v_{i-(n-1)}\right) \\ . \\ v_0 = \beta_0, v_1 = \beta_1, \dots, v_{n-1} = \beta_{n-1} \end{cases} \tag{4.12}$$

$$\begin{cases} v_{i+1}^{(k)} = c_{n-1}v_i + c_{n-2}v_{i-1} + \cdots + c_0 v_{i-(n-1)} + h\left(b_n f_2\left(t_{i+1}, w_{i+1}^{(k-1)}, v_{i+1}^{(k-1)}\right)\right) \\ \qquad + \cdots + b_0 f_2\left(t_{i-(n-1)}, w_{i-(n-1)}, v_{i-(n-1)}\right) \\ . \\ v_0 = \beta_0, v_1 = \beta_1, \dots, v_{n-1} = \beta_{n-1} \end{cases} \tag{4.13}$$

Now in continue we will discuss about the process of obtaining multi-step methods mentioned above briefly.

Example

In this section, the approximate solutions obtained from Predictor- Corrector methods are compared with exact solution by using numerical example.

Solve the following Vander pol's equation

$y'' - (0.1)(1-y^2)y' + y = 0$

Using Predictor- Corrector method for t=0.2 with the initial values, $y(0) = 1, y'(0) = 0$

Solution let

$$\frac{dy}{dt} = p = f_1(t, y, p)$$

Then $\frac{dy}{dt} = (0.1)(1-y^2)p - y = f_1(t, y, p)$

Thus, the given Vander pol's equation reduced to two first-order equations.

In the present problem, we are given that $t_0 = 0, y_0 = 1, p_0 = y_0' = 0$, taking h=0.1 we compute

With a single step, like fourth- order Runge- kutta method, approximate to $y(0.1), y'(0.1)$ to get. Then this approximation, respectively, with w_1, v_1 below, we show.

$$v_0 = y_0' = 0, w_0 = y_0 = 1$$

$$w_1 = w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}\big(0 + 2(-0.005) - 2(-0.005) + (-0.0995)\big) \simeq 0.98008$$

$$v_1 = v_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$= 0 + \frac{1}{6}\big(-0.1 + (-0.1) + (-0.0995) + (-0.0995)\big) = -0.665$$

Using two-step Adam- Bashforth, to get an initial approximation.Then, using a two- step Adams-Moulton, resulting answers are correct.

$$w_0 = 1, w_1 = 0.98$$

$$w_2^{(0)} = w_1 - \frac{h}{2}\big(f_1(t_0, w_0, v_0) + 3f_1(t_1, w_1, v_1)\big)$$

$$= 0.98 - \frac{0.1}{2}\big(-0.0665 + 3(-0.0665)\big) = 09933$$

$$v_2^{(0)} = v_1 - \frac{h}{2}\big(f_2(t_0, w_0, v_0) + 3f_2(t_1, w_1, v_1)\big)$$

$$= -0.0665 - \frac{0.1}{2}\Big((0-1) + 3\big(0.1(1-(0.98008)^2)\big)(-0.0665)(0.98008)\Big) = 0.1305$$

$$w_2^{(1)} = w_1 + \frac{h}{12}\Big(5f_1\left(t_2, w_2^{(0)}, v_2^{(0)}\right) + 8f_1(t_1, w_1, v_1) - f_1(t_0, w_0, v_0)\Big)$$

$$= 0.98008 + \frac{0.1}{12}(5(0.1305) + 8(-0.0665) - 0) = 0.9810$$

$$v_2^{(1)} = v_1 + \frac{h}{12}\Big(5f_2\left(t_2, w_2^{(0)}, v_2^{(0)}\right) + 8f_2(t_1, w_1, v_1) - f_2(t_0, w_0, v_0)\Big)$$

$$v_2^{(1)} = 0.0665 + \frac{0.1}{12}\big(5\big(0.1(1-(0.9933)^2)\big)(0.1305) - 0.9933\big)$$

$$+8(0.1(1-(0.98008)^2)(-0.0665) - 0.98008) - (0-1))$$

$$\rightarrow v_2^{(1)} = 0.0665 + \frac{0.1}{12}(-4.966 - 7.843 + 1) = -0.0319$$

$$\rightarrow v_2^{(1)} = -0.0319$$

CONCLUSION

In this research, we used Predictor- Corrector methods to solve high-order differential equations. In the example solved.

The Runge- kutta method to get the values w_1 and v_1 then using Adams- Bashforthe w_2 and v_2 are obtained then we modify the resulting solution with Adams- Moulton.

ACKNOWLEDGEMENT

This article has resulted from the research project supported by Islamic Azad University of Ayatollah Amoli Branch in Iran.

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