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Research Article

ON THE NON-HOMOGENEOUS SEXTIC EQUATION WITH FIVE UNKNOWNS $2(X-Y)(X^3+Y^3)=7(Z^2-W^2)P^4$

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ABSTRACT

The Sextic non-homogeneous equation with five unknowns represented by the Diophantine equation $2(X-Y)(X^3+Y^3)=7(Z^2-W^2)P^4$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special numbers are exhibited.

Keywords: *Integral solutions, Sextic non-homogeneous equation, Lattice points*
Mathematics Subject Classification: 11D41

INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems (Dickson, 1952; Carmichael, 1959; Mordell, 1969 and Telang, 1996) particularly, in (Gopalan and Sangeetha, 2010; Gopalan *et al.,* 2007;) Sextic equations with three unknowns are studied for their integral solutions. Gopalan and Vijayashankar (2010), Gopalan *et al.,* (2012), Gopalan *et al.,* (2012), Gopalan *et al.,* (2013), Gopalan *et al.,* (2013), Gopalan *et al.,* (2013) and Gopalan *et al.,* (2013) (please mention the year as 2012a, 2012b and so on. Please do the changes in the reference section also) analyze Sextic equations with four unknowns for their non-zero integer solutions. Gopalan *et al.,* (2012), [14, 15] analyze Sextic equations with five unknowns for their non-zero integer solutions. This communication analyzes a Sextic equation with five unknowns given by $2(X-Y)(X^3+Y^3)=7(Z^2-W^2)P^4$.

Infinitely many Quintuples (X, Y, Z, W, P) satisfying the above equation is obtained. Various interesting properties among the values of X, Y, Z, W and P are presented.

NOTATIONS USED:

- $F_{4,n,3}$ **:**Four dimensional Figurate number of rank n whose generating polygon is a triangle
- $F_{4,n,4}$ **:**Four dimensional Figurate number of rank n whose generating polygon is a square
- $F_{4,n,5}$ **:**Four dimensional Figurate number of rank n whose generating polygon is a pentagon
- P_n^m **:** Pyramidal number of rank n with m sides
- $T_{m,n}$ **:** Polygonal number of rank n with m sides
- CH_n **:** Centered Hexagonal number of rank n
- Pt_n **:** Pentatope number of rank n
- $CP_{6,n}$ **:** Centered hexagonal Pyramidal number of rank n
- $CP_{22,n}$ **:** Centered Icosidigonal Pyramidal number of rank n
- $Ct_{22,n}$ **:** Centered Icosidigonal number of rank n
- $Ct_{25,n}$ **:** Centered Icosipentagonal number of rank n
- Cs_n **:** Centered Square number of rank n

- $Tetra_n$ **:** Tetrahedral number of rank n
- Pr_n **:** Pronic number of rank n
- PP_n **:** Pentagonal Pyramidal number of rank n
- Pen_n **:** Pentagonal number of rank n
- So_n : Stella Octangular number of rank n
- CC_n : Centered Cube number of rank n
- Hex_n : Hexagonal number of rank n
- RD_n : Rhombic Dodecagonal number of rank n
- TO_n : Truncated Octahedral number of rank n
- HD_n : Hendecagonal number of rank n

MATERIALS AND METHODS

The Sextic equation with five unknowns to be solved is

$$2(X-Y)(X^3+Y^3)=7(Z^2-W^2)P^4 \quad (1)$$

The processes of obtaining patterns of integral solutions to (1) are illustrated below.

Introducing the transformations $X=u+v, Y=u-v, Z=2u+v, W=2u-v, u\neq v$ (2)

in (1), it is written as

$$u^2+3v^2=7P^4 \quad (3)$$

Assume $P(a,b)=a^2+3b^2,\ a,b\neq 0$ (4)

Here, we present four different choices of solutions of (3) and hence, obtain four different patterns of solutions to (1).

Pattern: 1

Write 7 as

$$7=(2+i\sqrt{3})(2-i\sqrt{3}) \quad (5)$$

Using (4) and (5) in (3) and employing the method of factorization, define

$$(u+i\sqrt{3}v)=(2+i\sqrt{3})(a+i\sqrt{3}b)^4 \quad (6)$$

Equating the real and imaginary parts on both sides, we get

$$u=2a^4+18b^4-36a^2b^2-12a^3b+36ab^3$$

$$v=a^4+9b^4-18a^2b^2+8a^3b-24ab^3$$

Substituting the values of u and v in (2), we get

$$\left.\begin{aligned} X&=X(a,b)=3a^4+27b^4-54a^2b^2-4a^3b+12ab^3\\ Y&=Y(a,b)=a^4+9b^4-18a^2b^2-20a^3b+60ab^3\\ Z&=Z(a,b)=5a^4+45b^4-90a^2b^2-16a^3b+48ab^3\\ W&=W(a,b)=3a^4+27b^4-54a^2b^2-32a^3b+96ab^3 \end{aligned}\right\} \quad (7)$$

Thus (4) and (7) represent the non-zero distinct integer solutions of (1).

Properties:

- $X(a,1)-36F_{4,a,4}+114Tetra_a+21T_{4,a}\equiv 1(\mathrm{mod}\,2)$

- $Y(1,b) - 216F_{4,b,3} - 6CP_{6,b} + 39CH_b \equiv 1 \pmod{19}$
- $Z(a,1) - 120Pt_a + 23SO_a + 145Pr_a \equiv 0 \pmod{5}$
- $W(a,1) - 24F_{4,a,5} + 84P_a^5 + 14Pen_a \equiv 0 \pmod{3}$
- $Z(1,b) - W(1,b) - 216F_{4,b,4} + 39CC_b + 297T_{4,b} \equiv -37 \pmod{41}$

Pattern: 2

Instead of (5), write 7 as

$$7 = \frac{(1+i3\sqrt{3})(1-i3\sqrt{3})}{4} \qquad (8)$$

Following the procedure presented in pattern: 1, the corresponding values of X, Y, Z and W satisfying (1) are

$$\left.\begin{aligned} X &= X(a,b) = 2a^4 - 36a^2b^2 + 18b^4 - 16a^3b + 48ab^3 \\ Y &= Y(a,b) = -a^4 + 18a^2b^2 - 9b^4 - 20a^3b + 60ab^3 \\ Z &= Z(a,b) = \frac{1}{2}[5a^4 - 90a^2b^2 + 45b^4 - 68a^3b + 204ab^3] \\ W &= W(a,b) = \frac{1}{2}[-a^4 + 18a^2b^2 - 9b^4 - 76a^3b + 228ab^3] \end{aligned}\right\} \qquad (9)$$

As our interest is on finding integer solutions, we choose a and b suitably so that the values of X, Y, Z, W and P are integers.

Replacing a by 2A and b by 2B in (4) and (9), the corresponding integer solutions of (1) in two parameters are

$$X = X(A,B) = 32A^4 - 576A^2B^2 + 288B^4 - 256A^3B + 768AB^3$$
$$Y = Y(A,B) = -16A^4 + 288A^2B^2 - 144B^4 - 320A^3B + 960AB^3$$
$$Z = Z(A,B) = 40A^4 - 720A^2B^2 + 360B^4 - 544A^3B + 1632AB^3$$
$$W = W(A,B) = -8A^4 + 144A^2B^2 - 72B^4 - 608A^3B + 1824AB^3$$
$$P = P(A,B) = 4A^2 + 12B^2$$

Properties:

- $X(A,1) - 768F_{4,A,3} + 896PP_A + 240Hex_A \equiv 0 \pmod{2}$
- $Y(A,1) + 384Pt_A + 112CC_A - 128T_{4,A} \equiv 0 \pmod{4}$
- $Z(A,1) - 480F_{4,A,4} + 372SO_A + 1040Pr_A \equiv 0 \pmod{5}$
- $W(A,1) + 192F_{4,A,3} + 560CP_{6,A} - 116Hex_A \equiv 0 \pmod{12}$
- $P(B,B)$ is a Nasty Number

Pattern: 3

(3) Can be written as $u^2 + 3v^2 = 7P^4 * 1$ (10)

Write 1 and 7 as

$$\left.\begin{aligned} 1 &= \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \\ 10 &= (2+i\sqrt{3})(2-i\sqrt{3}) \end{aligned}\right\} \qquad (11)$$

Substituting (4) and (11) in (10) and employing the method of factorization, define

Research Article

$$u+i\sqrt{3}v=\frac{(2+i\sqrt{3})(1+i\sqrt{3})}{2}(a+i\sqrt{3}b)^4$$

Following the procedure presented in pattern: 2, the corresponding values of X, Y, Z, W and P satisfying (1) are

$$X=X(A,B)=16A^4-288A^2B^2+144B^4-320A^3B+960AB^3$$

$$Y=Y(A,B)=-32A^4+576A^2B^2-288B^4-256A^3B+768AB^3$$

$$Z=Z(A,B)=8A^4-144A^2B^2+72B^4-608A^3B+1824AB^3$$

$$W=W(A,B)=-40A^4+720A^2B^2-360B^4-544A^3B+1632AB^3$$

$$P=P(A,B)=4A^2+12B^2$$

PROPERTIES:

- $X(A,1)-192F_{4,A,4}+800P_A^5+8CS_A\equiv 0(\mathrm{mod}\,4)$
- $Y(A,1)+768F_{4,A,3}-12PP_A-934T_{4,A}\equiv 3A(\mathrm{mod}\,8)$
- $Z(A,1)-192Pt_A+656CP_{6,A}+232\,Pr_A\equiv 0(\mathrm{mod}\,8)$
- $W(A,1)+960F_{4,A,3}+76RD_A+228Hex_A\equiv 0(\mathrm{mod}\,2)$
- $X(1,B)-1152F_{4,B,5}-30TO_B-990T_{4,B}\equiv 0(\mathrm{mod}\,10)$

PATTERN: 4

Instead of (11), Write 1 and 7 as

$$\left.\begin{aligned}1&=\frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}\\7&=\frac{(1+i3\sqrt{3})(1-i3\sqrt{3})}{4}\end{aligned}\right\}\qquad(12)$$

Following the procedure presented in pattern: 3, the corresponding values of X, Y, Z, W and P satisfying (1) are

$$\begin{aligned}X&=X(a,b)=-a^4-9b^4+18a^2b^2-20a^3b+60ab^3\\Y&=Y(a,b)=-3a^4-27b^4+54a^2b^2-4a^3b+12ab^3\\Z&=Z(a,b)=-3a^4-27b^4+54a^2b^2-32a^3b+96ab^3\\W&=W(a,b)=-5a^4-45b^4+90a^2b^2-16a^3b+48ab^3\end{aligned}\qquad(13)$$

Thus (4) and (13) represent the non-zero distinct integer solutions of (1).

Properties:

- $X(a,1)+24F_{4,a,3}+42P_a^4-4Ct_{25,a}\equiv -13(\mathrm{mod}\,23)$
- $Y(a,1)+24F_{4,a,5}-3CC_a-24CH_a\equiv -4(\mathrm{mod}\,7)$
- $W(a,1)-60F_{4,a,4}-18PP_a-11Ct_{22,a}\equiv 3(\mathrm{mod}\,53)$
- $Z(a,1)+24F_{4,a,5}+6CP_{22,a}-14HD_a\equiv 2a(\mathrm{mod}\,3)$
- $X(1,b)-Y(1,b)-432Pt_b+30SO_b+1224T_{4,b}\equiv 0(\mathrm{mod}\,2)$

CONCLUSION

First of all, it is worth to mention here that in (2), the values of Z and W may also be represented by $Z=2uv+1, W=2uv-1$ and thus will obtain other choices of solutions to (1). In conclusion, one may consider other forms of Sextic equation with five unknowns and search for their integer solutions.

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