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***Research Article***

# INTEGRAL SOLUTIONS OF $x^3+y^3+z^3=3xyz+14(x+y)w^3$

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**ABSTRACT**
The non-homogeneous quadratic equation with four unknowns represented by $x^3+y^3+z^3=3xyz+14(x+y)w^3$ is analyzed for finding its non-zero distinct integral solutions. Four different methods have been presented for determining the integral solutions of the non-homogeneous bi-quadratic equation under consideration. Employing the integral solutions of the above equation, a few interesting relations between special numbers are exhibited.
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## INTRODUCTION
The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, bi-quadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity (Dickson, 1952; Mordell, 1969; Carmichael, 1959; Telang, 1996; Smart, 1999). In this context, one may refer Gopalan and Pandichelvi (2009); Gopalan and Shanmuganandham (2010); Gopalan and Padma (2010); Gopalan and Sangeetha (2011); Manju *et al.*, (2011); Manju *et al.*, (2012); Gopalan and Shanmuganandham (2012); Meena *et al.*, (2014); Gopalan *et al.*, (2014) for various problems on the bi-quadratic diophantine equations with four variables. However, often we come across non-homogeneous bi-quadratic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous equaton with four unknowns given by $x^3+y^3+z^3=3xyz+14(x+y)w^3$. A few relations among the solutions are presented.

***Notations***

$$t_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right).$$

$$P_n^m=\left(\frac{n(n+1)}{6}\right)[(m-2)n+(5-m)]$$

$$Pt_n=\frac{n(n+1)(n+2)(n+3)}{24}$$

$$SO_n=n(2n^2-1)$$

$$S_n=6n(n-1)+1$$

$$\Pr_n=n(n+1)$$

***Method of Analysis***

The bi-quadratic Diophantine equation with four unknowns to be solved for its non-zero distinct integral solution is

$$x^3+y^3+z^3=3xyz+14(x+y)w^3 \tag{1}$$

Different patterns of integer solutions to (1) are illustrated below:

***Pattern I***

Introducing the linear transformations

$$x = u + v, y = u - v, z = 2u \tag{2}$$

in (1), it is written as

$$u^2 + 3v^2 = 7w^3 \tag{3}$$

write 7 as

$$7 = (2 + i\sqrt{3})(2 - i\sqrt{3}) \tag{4}$$

Assume

$$w = a^2 + 3b^2 \tag{5}$$

where a and b are non-zero distinct integers.
Using (4) and (5) in (3) and employing the method of factorization, define

$$u + i\sqrt{3}v = (2 + i\sqrt{3})(a + i\sqrt{3}b)^3$$

Equating real and imaginary parts in the above equation, we get,

$$u = 2a^3 - 18ab^2 - 9a^2b + 9b^3$$
$$v = a^3 - 9ab^2 + 6a^2b - 6b^3$$

Substituting the values of u, v in (2),we have

$$\begin{aligned} x(a,b) &= 3a^3 - 27ab^2 - 3a^2b + 3b^3 \\ y(a,b) &= a^3 - 9ab^2 - 15a^2b + 15b^3 \\ z(a,b) &= 4a^3 - 36ab^2 - 18a^2b + 18b^3 \end{aligned} \tag{6}$$

Thus (5) and (6) represent the non-zero distinct integer solutions of (1).

***Properties***

- $2x(a+1,a) + 2y(a+1,a) + 108t_{4,a} + 24t_{3,a} - 8$ is a cubical integer.
- $z(a,1) + w(a,1) - 8P_a^5 + 42t_{3,a} \equiv 0 (\bmod 3)$
- $w(1,2^n) - 3Ky_n$ is divisible by 2.
- $y(a,1) + z(a,1) - 10P_a^5 + 6S_a + 24t_{4,a}$ is divisible by 3.
- $x(1,b) + w(1,b) - 6P_b^5 + 6t_{3,b} + 24t_{4,b} = 1$
- $z(2^n,1) - 4y(2^n,1) - 42Ky_n \equiv 0 (\bmod 2)$
- $z(a,1) - 14y(a,1) - 18w(a,1) + 45$ is a nasty number.
- $x(1,b) - 3y(1,b) + 21So_b \equiv 0 (\bmod 21)$
- $7[z(1,b) - 4y(1,b) + 63OH_b]$ is a perfect square.

***Pattern II***

(3) can be written as

$$u^2 + 3v^2 = 7 * 1 * w^3 \tag{7}$$

Write 1 as

$$1 = \frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} \tag{8}$$

Substituting (4), (5) and (8) in (7), proceeding as in Pattern I, the non-zero distinct integral solutions to (1) are given by

$$x(a,b) = a^3 - 9ab^2 - 15a^2b + 15b^3$$

$$y(a,b) = -2a^3 + 18ab^2 - 12a^2b + 12b^3$$
$$z(a,b) = -a^3 + 9ab^2 - 27a^2b + 27b^3$$
$$w(a,b) = a^2 + 3b^2$$

***Properties***

- $2x(a,a+1) + y(a,a+1) - 84P_{ra} - 84t_{3,a} + 42t_{4,a} = 42$
- $y(b+1,b) - 2z(b+1,b) - 42P_{rb} - 42t_{4,b} = 0$
- $y(1,b) - x(1,b) - 3z(1,b) - 42P_{rb} + 42t_{4,b} + 42So_b = 0$
- $y(a,1) - 2z(a,1) + 12w(a,1) + 6$ is a nasty number.
- $x(2^n,1) + z(2^n,1) + w(2^n,1) + 123J_{2n} = 4$

***Pattern III***

Using the linear transformations

$$x = 3(\alpha+\beta), y = 3(\alpha-\beta), z = 6\alpha, w = 3\gamma \tag{9}$$

in (1), it leads to

$$\alpha^2 + 3\beta^2 = 21\gamma^3 \tag{10}$$

write 21 as

$$21 = (3+i2\sqrt{3})(3-i2\sqrt{3}) \tag{11}$$

Assume

$$\gamma = R^2 + 3S^2 \tag{12}$$

substituting (11) and (12) in (10) and employing the method of factorization, define

$$\alpha + i\sqrt{3}\beta = (3+i2\sqrt{3})(R+iS\sqrt{3})^3$$

Equating the real and imaginary parts, we get

$$\alpha = 3R^3 - 27RS^2 - 18R^2S + 18S^3$$
$$\beta = 2R^3 - 18RS^2 + 9R^2S - 9S^3$$

Using these values of $\alpha$ and $\beta$ in (9), the non-zero distinct integral solutions to (1) are obtained as

$$x(R,S) = 15R^3 - 135RS^2 - 27R^2S + 27S^3$$
$$y(R,S) = 3R^3 - 27RS^2 - 81R^2S + 81S^3$$
$$z(R,S) = 18R^3 - 162RS^2 - 108R^2S + 108S^3$$
$$w(R,S) = 3R^2 + 9S^2$$

***Properties***

- $x(R,1) - 5y(R,1) + 21w(R,1) + 189$ is a perfect square.
- $x(2^n,1) - 5y(2^n,1) + w(2^n,1) - 1143J_{2n} = 12$
- $z(R,1) - 6y(R,1) + 36w(R,1) + 54$ is a nasty number
- $6y(1,S) - z(1,S) + 42w(1,s) - 756P_S^5 + 378P_{rS} - 378t_{4,S} = 126$
- $y(R,1) + w(R,1) - 6P_R^5 + 162t_{3,R} \equiv 0 \pmod 3$

***Pattern IV***

In (11), 21 can also be written as

$$21 = \frac{(3+5i\sqrt{3})(3-5i\sqrt{3})}{4} \qquad (13)$$

Using (13) and (12) in (11) and proceeding as in pattern III, we get the non-zero distinct integral solutions to (1) be

$$\begin{aligned} x(R,S) &= 12R^3 - 108RS^2 - 54R^2S + 54S^3 \\ y(R,S) &= -3R^3 + 27RS^2 - 81R^2S + 81S^3 \\ z(R,S) &= 9R^3 - 81RS^2 - 135R^2S + 135S^3 \\ w(R,S) &= 3R^2 + 9S^2 \end{aligned} \qquad (14)$$

***Properties***

- $x(R,1) + 4y(R,1) + 126w(R,1) + 24$ is a nasty number.
- $z(R,1) + 3y(R,1) - 72w(R,1) + 216t_{3,R}$ is an even number.
- $9x(1,S) - 12z(1,S) - 1134P_{rS} + 2268P_S^5 = 0$
- $z(1,S) + 3y(1,S) + 42w(1,S) - 756P_S^5 + 378P_{rS} - 378t_{4,S} = 126.$
- $z(2^n,1) + 3y(2^n,1) + 378j_{2n} \equiv 0 \pmod 2$

**CONCLUSION**

In this paper, we have presented four different patterns of integer solutions to the bi-quadratic with four unknowns given by $x^3 + y^3 + z^3 = 3xyz + 14(x+y)w^3$. It is to be noted that, in addition to (4) in pattern I and (8) in pattern II, we have the following choices :

$$7 = \frac{(1+i3\sqrt{3})(1-i3\sqrt{3})}{4}$$

$$1 = \frac{(1+4i\sqrt{3})(1-4i\sqrt{3})}{49}$$

and $1 = \dfrac{(1+15i\sqrt{3})(1-15i\sqrt{3})}{26}$

Following the procedure presented in this paper, one may obtain other choices of integer solutions to the equation under consideration. To conclude, one may search for the integral solutions of the bi-quadratic equation with four or more variables.

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