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Research Article

CERTAIN INTEGRAL PROPERTIES OF GENERALIZED CLASS OF POLYNOMIALS AND GENERALIZED CONTOUR INTEGRAL ASSOCIATED WITH FEYNMAN INTEGRALS

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ABSTRACT
The object of the present paper is to discuss certain integral properties of a general class of polynomials and I-function, proposed by Inayat-Hussain which contains a certain class of Feynman integrals. We establish certain new double integral relations pertaining to a product involving general class of polynomials and I-function. These double integral relations are unified in nature and act as key formulae from which we can obtain as their special cases, double integral relations concerning a large number of simpler special function and polynomials. For the sake of illustration, we record here some special cases of our main results which are also new and of interest by themselves. The results established here are basic in nature and are likely to find useful applications in several fields.
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Key Words: *Feynman Integrals, I-Function, General Class Of Polynomials*

INTRODUCTION
The I-function will be defined and represented as follows [2]

$$I^{m,n}_{p_i,q_i;r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \phi(\xi)z^{\xi}d\xi \tag{1}$$

where

$$\phi(\xi)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j-\beta_j\xi\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\alpha_j\xi\right)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}+\beta_{ji}\xi\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}-\alpha_{ji}\xi\right)\right]} \tag{2}$$

and m,n,p_i,q_i are integers satisfy $0\le n\le p_i, 1\le m\le q_i\left(i=1,...,r\right)$, r is finite, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}$ are positive numbers and a_j,b_j,a_{ji},b_{ji} are complex numbers. I-function which is a generalized form of the well known Fox's H-function [4, p.10, Eqn.(2.1.1)]. In the sequel the I-function will be studied under the following conditions of existence:

where

(I) $$A_i>0,\left|\arg z\right|<\frac{A_i\pi}{2} \tag{3}$$

(II) $$A_i\ge 0,\left|\arg z\right|\le\frac{A_i\pi}{2} \text{ and } \operatorname{Re}(B+1)<0 \tag{4}$$

where

$$A_i = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p_i} \alpha_{ji} + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q_i} \beta_{ji}, \forall i = (1,2,...,r) \tag{5}$$

and

$$B = \sum_{j=1}^{m} b_j + \sum_{j=m+1}^{q_i} b_{ji} - \sum_{j=1}^{n} a_j - \sum_{j=n+1}^{p_i} a_{ji} + \frac{1}{2}(p_i - q_i), \forall i = (1,2,...r) \tag{6}$$

The general class of polynomials $S_{n_1,...,n_r}^{m_1,...,m_r}[x]$ will be defined and represented as follows [3, p.185, Eqn. (7)]:

$$S_{n_1,...,n_r}^{m_1,...,m_r}[x] = \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} x^{l_i} \tag{7}$$

where $n_1,...,n_r = 0,1,2,...; m_1,...,m_r$ are arbitrary positive integers, the coefficients $A_{n_i,l_i}(n_i,l_i \geq 0)$ are arbitrary constants, real or complex. $S_{n_1,...,n_r}^{m_1,...,m_r}[x]$ yeilds a number of known polynomials as its special cases. These includes, among other, the Jacobi polynomials, the Bessel Polynomials, the Hermite Polynomials, the Lagurre Polynomials, the Brafman Polynomials and several others [5, p. 158-161].

Main Results

We shall establish the following results:

(A)

$$\int_0^1\int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^a \left[\frac{1-y}{1-xy}\right]^b \left[\frac{1-xy}{(1-x)(1-y)}\right] S_{n_1,...,n_r}^{m_1,...,m_r}\left[\frac{(1-x)ty}{1-xy}\right] I_{p_i,q_i;r}^{m,n}\left[\frac{(1-y)t}{1-xy}\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dxdy$$

$$= \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} t^{l_i} \Gamma(a+l_i) I_{p_i+1,q_i+1;r}^{m,n+1}\left[z\left|\begin{matrix}(1-b,1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i},(1-a-b-l_i,1)\end{matrix}\right.\right] \tag{8}$$

The above result is valid under the conditions (3), (4); $\mathrm{Re}(a+b+b_j/\beta_j) > 0 (1 \leq j \leq m)$ and a,b are positive. Also $0 < x < 1$ and $0 < y < 1$.

Proof of the above result- In the left hand side of equation (8) put the value of $I_{p_i,q_i;r}^{m,n}[x]$ and $S_{n_1,...,n_r}^{m_1,...,m_r}[x]$ from (1) and (7) respectively, interchanging the order of integration and summation then making the use of known result [1, p.145], we get the result (8) after little simplification.

(B)

$$\int_0^\infty\int_0^\infty \phi(s+t) t^{b-1} s^{a-1} S_{n_1,...,n_r}^{m_1,...,m_r}[s] I_{p_i,q_i;r}^{m,n}\left[t\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dsdt$$

$$= \sum_{l_1=0}^{[n_1/m_1]} \cdots \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} t^{l_i} \Gamma(a+l_i) \int_0^\infty \phi(z) z^{a+b+l_i-1} \times$$

$$I_{p_i+1,q_i+1;r}^{m,n+1}\left[z\left|\begin{matrix}(1-b,1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i},(1-a-b-l_i,1)\end{matrix}\right.\right]dz \tag{9}$$

The above result is valid under the conditions (3), (4); $\mathrm{Re}(a+b+b_j/\beta_j) > 0 (1 \leq j \leq m)$ and a,b are positive. Also $0 < s < \infty$ and $0 < t < \infty$.

Proof of the above result: In the left hand side of equation (9) put the value of $I_{p_i,q_i;r}^{m,n}[x]$ and $S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[x]$ from (1) and (7) respectively, interchanging the order of integration and summation then making the use of known result [1, p.177], we get the result (9) after little simplification.

(C)

$$\int_0^1\int_0^1 f(st)(1-s)^{a-1}(1-t)^{b-1}t^a S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[t(1-s)] I_{p_i,q_i;r}^{m,n}\left[(1-t)\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dsdt$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}t^{l_i}\Gamma(a+l_i)\int_0^1 f(z)(1-z)^{a+b+l_i-1}\times$$

$$I_{p_i+1,q_i+1;r}^{m,n+1}\left[(1-z)\left|\begin{matrix}(1-b,1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i},(1-a-b-l_i,1)\end{matrix}\right.\right]dz \tag{10}$$

The above result is valid under the conditions (3), (4); $\mathrm{Re}\left(a+b+b_j/\beta_j\right)>0\,(1\le j\le m)$ and a,b are positive. Also $0<s<1$ and $0<t<1$.

Proof of the above result: In the left hand side of equation (10) put the value of $I_{p_i,q_i;r}^{m,n}[x]$ and $S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[x]$ from (1) and (7) respectively, interchanging the order of integration and summation then making the use of known result [1, p.243], we get the result (10) after little simplification.

(D)

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{a+\sigma}\left[\frac{1-y}{1-xy}\right]^{b}\frac{1}{(1-x)}S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[\frac{(1-x)y}{1-xy}\right]I_{p_i,q_i;r}^{m,n}\left[\frac{(1-y)ty}{1-xy}\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dxdy$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\Gamma(b+1)I_{p_i+1,q_i+1;r}^{m,n+1}\left[t\left|\begin{matrix}(1-a-\sigma-l_i,1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i},(-a-b-\sigma-l_i,1)\end{matrix}\right.\right] \tag{11}$$

The above result is valid under the conditions (3), (4); $\mathrm{Re}\left(a+b+\sigma+b_j/\beta_j\right)>0\,(1\le j\le m)$ and a,b,σ are positive. Also $0<x<1$ and $0<y<1$.

***Proof of the above result*:** In the left hand side of equation (11) put the value of $I_{p_i,q_i;r}^{m,n}[x]$ and $S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[x]$ from (1) and (7) respectively, interchanging the order of integration and summation then making the use of known result [1, p.145], we get the result (11) after little simplification.

Special Cases

(I) By applying the our results given in (A), (B), (C) and (D) to the case of Hermite polynomials [5] by setting $S_n^2(x)\to x^{n/2}H_n\left[\frac{1}{2\sqrt{x}}\right]$ in which $m_1,\dots,m_r=2;\ n_1,\dots,n_r=n;\ r=1;\ A_{n_i,l_i}=(-1)^l$, we have the following interesting results.

(A1)

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{a}\left[\frac{1-y}{1-xy}\right]^{b}\left[\frac{1-xy}{(1-x)(1-y)}\right]\left[\frac{(1-x)ty}{1-xy}\right]^{n/2}H_n\left[\frac{1}{2\sqrt{\frac{(1-x)ty}{1-xy}}}\right]I_{p_i,q_i;r}^{m,n}\left[\frac{(1-y)t}{1-xy}\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dxdy$$

$$= \sum_{l=0}^{[n/2]} \frac{(-n)_{2l}}{l!} (-1)^l t^l \Gamma(a+l) I^{m,n+1}_{p_i+1,q_i+1;r}\left[t \left| \begin{matrix} (1-b,1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i},(1-a-b-l,1) \end{matrix} \right. \right] \tag{12}$$

The conditions of convergence of the above result can be easily obtained from those of (8)

(B1)

$$\int_0^\infty \int_0^\infty \phi(s+t) t^{b-1} s^{a+n/2-1} H_n\left[\frac{1}{2\sqrt{s}}\right] I^{m,n}_{p_i,q_i;r}\left[t \left| \begin{matrix} (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] dsdt$$

$$= \sum_{l=0}^{[n/2]} \frac{(-n)_{2l}}{l!} (-1)^l \Gamma(a+l) \int_0^\infty \phi(z) z^{a+b+l-1} I^{m,n+1}_{p_i+1,q_i+1;r}\left[z \left| \begin{matrix} (1-b,1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i},(1-a-b-l,1) \end{matrix} \right. \right] dz \tag{13}$$

The conditions of convergence of the above result can be easily obtained from those of (9)

(C1)

$$\int_0^1 \int_0^1 f(st)(1-s)^{a+n/2-1}(1-t)^{b-1} t^{a+n/2} H_n\left[\frac{1}{2\sqrt{t(1-s)}}\right] I^{m,n}_{p_i,q_i;r}\left[(1-t) \left| \begin{matrix} (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] dsdt$$

$$= \sum_{l=0}^{[n/2]} \frac{(-n)_{2l}}{l!} (-1)^l \Gamma(a+l) \int_0^1 f(z)(1-z)^{a+b+l-1} I^{m,n+1}_{p_i+1,q_i+1;r}\left[(1-z) \left| \begin{matrix} (1-b,1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i},(1-a-b-l,1) \end{matrix} \right. \right] dz \tag{14}$$

The conditions of convergence of the above result can be easily obtained from those of (10)

(D1)

$$\int_0^1 \int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^{a+n/2+\sigma} \left[\frac{1-y}{1-xy}\right]^b \frac{1}{(1-x)} H_n\left[\frac{1}{2\sqrt{\frac{(1-x)y}{1-xy}}}\right] I^{m,n}_{p_i,q_i;r}\left[\frac{(1-y)ty}{1-xy} \left| \begin{matrix} (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] dxdy$$

$$= \sum_{l=0}^{[n/2]} \frac{(-n)_{2l}}{l!} (-1)^l \Gamma(b+1) I^{m,n+1}_{p_i+1,q_i+1;r}\left[t \left| \begin{matrix} (1-a-\sigma-l,1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i},(-a-b-\sigma-l,1) \end{matrix} \right. \right] \tag{15}$$

The conditions of convergence of the above result can be easily obtained from those of (11)

(II) By applying the our results given in (A), (B), (C) and (D) to the case of Laguerre polynomials [5] by setting $S_n^2(x) \to L_n^{(\alpha)}[x]$ in which $m_1,...,m_r = 1;\ n_1,...,n_r = n; r = 1;\ A_{n_i,l_i} = \binom{n+\alpha}{n}\frac{1}{(\alpha+1)_l}$, we have the following interesting results.

(A2)

$$\int_0^1 \int_0^1 \left[\frac{(1-x)y}{1-xy}\right]^a \left[\frac{1-y}{1-xy}\right]^b \left[\frac{1-xy}{(1-x)(1-y)}\right] L_n^{(\alpha)}\left[\frac{(1-x)ty}{1-xy}\right] I^{m,n}_{p_i,q_i;r}\left[\frac{(1-y)t}{1-xy} \left| \begin{matrix} (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] dxdy$$

$$= \sum_{l=0}^{n} \frac{(-n)_l}{l!} \binom{n+\alpha}{n} \frac{1}{(\alpha+1)_l} t^l \Gamma(a+l) I^{m,n+1}_{p_i+1,q_i+1;r}\left[t \left| \begin{matrix} (1-b,1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i},(1-a-b-l,1) \end{matrix} \right. \right] \tag{16}$$

The conditions of convergence of the above result can be easily obtained from those of (8)

(B2)

$$\int_0^\infty\int_0^\infty \phi(s+t)t^{b-1}s^{a-1}L_n^{(\alpha)}[s]I_{p_i,q_i;r}^{m,n}\left[t\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dsdt$$

$$=\sum_{l=0}^{n}\frac{(-n)_l}{l!}\binom{n+\alpha}{n}\frac{1}{(\alpha+1)_l}\Gamma(a+l)\int_0^\infty \phi(z)z^{a+b+l-1}I_{p_i+1,q_i+1;r}^{m,n+1}\left[z\left|\begin{matrix}(1-b,1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i},(1-a-b-l,1)\end{matrix}\right.\right]dz \quad (17)$$

The conditions of convergence of the above result can be easily obtained from those of (9)

(C2)

$$\int_0^1\int_0^1 f(st)(1-s)^{a-1}(1-t)^{b-1}t^aL_n^{(\alpha)}[t(1-s)]I_{p_i,q_i;r}^{m,n}\left[(1-t)\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dsdt$$

$$=\sum_{l=0}^{n}\frac{(-n)_l}{l!}\binom{n+\alpha}{n}\frac{1}{(\alpha+1)_l}\Gamma(a+l)\int_0^1 f(z)(1-z)^{a+b+l-1}$$

$$I_{p_i+1,q_i+1;r}^{m,n+1}\left[(1-z)\left|\begin{matrix}(1-b,1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i},(1-a-b-l,1)\end{matrix}\right.\right]dz \quad (18)$$

The conditions of convergence of the above result can be easily obtained from those of (10)

(D2)

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{a+\sigma}\left[\frac{1-y}{1-xy}\right]^b\frac{1}{(1-x)}L_n^{(\alpha)}\left[\frac{(1-x)y}{1-xy}\right]I_{p_i,q_i;r}^{m,n}\left[\frac{(1-y)ty}{1-xy}\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dxdy$$

$$=\sum_{l=0}^{n}\frac{(-n)_l}{l!}\binom{n+\alpha}{n}\frac{1}{(\alpha+1)_l}\Gamma(b+1)I_{p_i+1,q_i+1;r}^{m,n+1}\left[t\left|\begin{matrix}(1-a-\sigma-l,1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i},(-a-b-\sigma-l,1)\end{matrix}\right.\right] \quad (19)$$

The conditions of convergence of the above result can be easily obtained from those of (11)

(III) If we put $r=1$, I-function reduces to the familiar Fox's H-function [4, p.10, Eqn. (2.1.1)], then the results (A), (B), (C) and (D) reduces to the following form:

(A3)

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^a\left[\frac{1-y}{1-xy}\right]^b\left[\frac{1-xy}{(1-x)(1-y)}\right]S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[\frac{(1-x)ty}{1-xy}\right]H_{p,q}^{m,n}\left[\frac{(1-y)t}{1-xy}\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\end{matrix}\right.\right]dxdy$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}t^{l_i}\Gamma(a+l_i)H_{p+1,q+1}^{m,n+1}\left[z\left|\begin{matrix}(1-b,1),(a_j,\alpha_j)_p\\(b_j,\beta_j)_{1,q},(1-a-b-l_i,1)\end{matrix}\right.\right] \quad (20)$$

The conditions of convergence of the above result can be easily obtained from those of (8)

(B3)

$$\int_0^\infty\int_0^\infty \phi(s+t)t^{b-1}s^{a-1}S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[s]H_{p,q}^{m,n}\left[t\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\end{matrix}\right.\right]dsdt$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}\Gamma(a+l_i)\int_0^\infty \phi(z)z^{a+b+l_i-1}H_{p+1,q+1}^{m,n+1}\left[z\left|\begin{matrix}(1-b,1),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(1-a-b-l_i,1)\end{matrix}\right.\right]dz \quad (21)$$

The conditions of convergence of the above result can be easily obtained from those of (9)

(C3)

$$\int_0^1\int_0^1 f(st)(1-s)^{a-1}(1-t)^{b-1}t^a S_{n_1,...,n_r}^{m_1,...,m_r}\left[t(1-s)\right]H_{p,q}^{m,n}\left[(1-t)\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\end{matrix}\right]dsdt$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\Gamma(a+l_i)\int_0^1 f(z)(1-z)^{a+b+l_i-1}H_{p+1,q+1}^{m,n+1}\left[(1-z)\middle|\begin{matrix}(1-b,1),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(1-a-b-l_i,1)\end{matrix}\right]dz \tag{22}$$

The conditions of convergence of the above result can be easily obtained from those of (10)

(D3)

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{a+\sigma}\left[\frac{1-y}{1-xy}\right]^{b}\frac{1}{(1-x)}S_{n_1,...,n_r}^{m_1,...,m_r}\left[\frac{(1-x)y}{1-xy}\right]H_{p,q}^{m,n}\left[\frac{(1-y)ty}{1-xy}\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\end{matrix}\right]dxdy$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\Gamma(b+1)H_{p+1,q+1}^{m,n+1}\left[t\middle|\begin{matrix}(1-a-\sigma-l_i,1),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)q,(-a-b-\sigma-l_i,1)\end{matrix}\right] \tag{23}$$

The conditions of convergence of the above result can be easily obtained from those of (11)

(IV) If we put $r=1, n=p_i=0, m=1, q_i=2, b_1=0, \beta_1=1, b_{m+1,1}=-\lambda, \beta_{m+1,1}=\mu$, then I-function reduces to the Wright's generalized Bessal function [6, p.257], i.e. $I_{0,2;1}^{1,0}\left[z\middle|\begin{matrix}(...)\\(0,1),(-\lambda,\mu)\end{matrix}\right]=J_\lambda^\mu(z)$ then results (A), (B), (C) and (D) reduces to the following form:

(A4)

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{a}\left[\frac{1-y}{1-xy}\right]^{b}\left[\frac{1-xy}{(1-x)(1-y)}\right]S_{n_1,...,n_r}^{m_1,...,m_r}\left[\frac{(1-x)ty}{1-xy}\right]J_\lambda^\mu\left(\frac{(1-y)t}{1-xy}\right)dxdy$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}t^{l_i}\sum_{n=0}^{\infty}\frac{(-t)^n}{\Gamma(\mu n+\lambda+1)}B(a+l_i,b+n) \tag{24}$$

The conditions of convergence of the above result can be easily obtained from those of (8)

(B4)

$$\int_0^\infty\int_0^\infty\phi(s+t)t^{b-1}s^{a-1}S_{n_1,...,n_r}^{m_1,...,m_r}[s]J_\lambda^\mu(t)dsdt$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\mu n+\lambda+1)}B(a+l_i,b+n)\int_0^\infty\phi(z)z^{a+b+l_i+n-1}dz \tag{25}$$

The conditions of convergence of the above result can be easily obtained from those of (9)

(C4)

$$\int_0^1\int_0^1 f(st)(1-s)^{a-1}(1-t)^{b-1}t^a S_{n_1,...,n_r}^{m_1,...,m_r}\left[t(1-s)\right]J_\lambda^\mu(1-t)dsdt$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\mu n+\lambda+1)}B(a+l_i,b+n)\int_0^1 f(z)(1-z)^{a+b+n+l_i-1}dz \tag{26}$$

The conditions of convergence of the above result can be easily obtained from those of (10)

(D4)

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{a+\sigma}\left[\frac{1-y}{1-xy}\right]^{b}\frac{1}{(1-x)}S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[\frac{(1-x)y}{1-xy}\right]J_{\lambda}^{\mu}\left(\frac{(1-y)ty}{1-xy}\right)dxdy$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\sum_{n=0}^{\infty}\frac{(-t)^n}{\Gamma(\mu n+\lambda+1)}B(a+\sigma+n+l_i,b+1) \qquad (27)$$

The conditions of convergence of the above result can be easily obtained from those of (11)

(V) If we put $r=1,n=p_i=p,m=1,q_i=q+1,b_1=0,\beta_1=1,a_j=1-a_j,b_{ji}=1-b_j,\beta_{ji}=\beta_j,$, then I-function reduces to the generalized wright hypergeometric function [7, p.287], i.e. $I_{p,q+1;1}^{1,p}\left[z\left|\begin{matrix}(1-a_j,\alpha_j)_{1,p}\\(0,1),(1-b_j,\beta_j)_{1,q}\end{matrix}\right.\right]={}_p\psi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix};-z\right]$ then results (A), (B), (C) and (D) reduces to the following form:

(A5)

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{a}\left[\frac{1-y}{1-xy}\right]^{b}\left[\frac{1-xy}{(1-x)(1-y)}\right]S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[\frac{(1-x)ty}{1-xy}\right]{}_p\psi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix};-\left(\frac{(1-y)t}{1-xy}\right)\right]dxdy$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}t^{l_i}\Gamma(a+l_i)\;{}_{p+1}\psi_{q+1}\left[\begin{matrix}(1-b,1);(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q};(1-a-b-l_i,1)\end{matrix};-t\right] \qquad (28)$$

The conditions of convergence of the above result can be easily obtained from those of (8)

(B5)

$$\int_0^{\infty}\int_0^{\infty}\phi(s+t)t^{b-1}s^{a-1}S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[s]\,{}_p\psi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix};-t\right]dsdt$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\Gamma(a+l_i)\int_0^{\infty}\phi(z)z^{a+b+l_i-1}\;{}_{p+1}\psi_{q+1}\left[\begin{matrix}(1-b,1);(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q};(1-a-b-l_i,1)\end{matrix};-z\right]dz \qquad (29)$$

The conditions of convergence of the above result can be easily obtained from those of (9)

(C5)

$$\int_0^1\int_0^1 f(st)(1-s)^{a-1}(1-t)^{b-1}t^aS_{n_1,\dots,n_r}^{m_1,\dots,m_r}[t(1-s)]\,{}_p\psi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix};-(1-t)\right]dsdt$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\Gamma(a+l_i)\int_0^1 f(z)(1-z)^{a+b+l_i-1}\;{}_{p+1}\psi_{q+1}\left[\begin{matrix}(1-b,1);(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q};(1-a-b-l_i,1)\end{matrix};z-1\right]dz \qquad (30)$$

The conditions of convergence of the above result can be easily obtained from those of (10)

(D5)

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{a+\sigma}\left[\frac{1-y}{1-xy}\right]^{b}\frac{1}{(1-x)}S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[\frac{(1-x)y}{1-xy}\right]{}_p\psi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix};-\frac{(1-y)ty}{1-xy}\right]dxdy$$

$$=\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\Gamma(b+1)\;{}_{p+1}\psi_{q+1}\left[\begin{matrix}(1-a-b-\sigma-l_i,1);(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q};(1-a-b-\sigma-l_i,1)\end{matrix};-t\right] \quad (31)$$

The conditions of convergence of the above result can be easily obtained from those of (11)

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