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Research Article

SOME NOTES ON ABSOLUTE SUMMABILITY FACTORS

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ABSTRACT
In this paper we give improvement to the result of [3] concerning absolute summability of infinite series.
2000 (MSC): 40F05, 40D25.

***Key Words**: Absolute Summability, Summability Factor*

INTRODUCTION

Let T be a lower triangular matrix, (s_n) a sequence, and

$$T_n := \sum_{v=0}^{n} t_{nv} s_v. \qquad (1.1)$$

A series $\sum a_n$ is said to be summable $|T|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta T_{n-1}|^k < \infty. \qquad (1.2)$$

Given any lower triangular matrix T one can associate the matrices $\overline{T}$ and $\hat{T}$, with entries defined by

$$\bar{t}_{nv} = \sum_{i=v}^{n} t_{ni}, \quad n, i = 0,1,2..., \qquad \hat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v}$$

respectively. With $s_n = \sum_{i=0}^{n} a_i \lambda_i$,

$$t_n = \sum_{v=0}^{n} t_{nv} s_v = \sum_{v=0}^{n} t_{nv} \sum_{i=0}^{v} a_i \lambda_i = \sum_{i=0}^{n} a_i \lambda_i \sum_{v=i}^{n} t_{nv} = \sum_{i=0}^{n} \bar{t}_{ni} a_i \lambda_i . \qquad (1.3)$$

$$Y_n := t_n - t_{n-1} = \sum_{i=0}^{n} \bar{t}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \bar{t}_{n-1,i} a_i \lambda_i = \sum_{i=0}^{n} \hat{t}_{ni} a_i \lambda_i, \quad as\ \bar{t}_{n-1,n} = 0. \qquad (1.4)$$

We call T a triangle if T is lower triangular and $t_{nn} \neq 0$ for all n.
We assume that (p_n) is a sequence of positive real numbers such that

$$P_n = p_0 + p_1 + ... + p_n \to \infty \ \ as \ \ n \to \infty.$$

In the special case when $t_{nv} = p_v / P_n$, summability $|T|_k$ reduces to $|\overline{N}, p_n|_k$ summability.

Rhoads and Savas (2005) proved the following result

Theorem 1.1. *Let A be a triangle with nonnegative entries satisfying*

(i) $\bar{a}_{n0} = 1$, $n = 0,1,...,$

(ii) $a_{n-1,v} \geq a_{nv}$ *for* $n \geq v+1$,

(iii) $na_{nn} = O(1)$, $1 = O(na_{nn})$,

(iv) $\sum_{v=0}^{n} a_{vv}\left|\hat{a}_{n,v+1}\right| = O(a_{nn})$.

If (X_n) *is a positive monotonic nondecreasing sequence such that*

(v) $|\lambda_n| X_n = O(1)$,

(vi) $\sum_{n=1}^{m} nX_n\left|\Delta^2\lambda_n\right| = O(1)$,

(vii) $\sum_{n=1}^{m} a_{nn}|t_n|^k = O(X_m)$, *where* $t_n = \dfrac{1}{n+1}\sum_{k=1}^{n} ka_k$,

then the series $\sum a_n\lambda_n$ *is summable* $|A|_k$, $k \geq 1$.

The object of this paper is to give two improvements to theorem 1.1 as follows
1. By weakening the condition (v) ,
2. By weakening the condition (vii),

Main Result

In what follows we prove the following

Theorem 2.1. *Let A be a triangle with nonnegative entries satisfying*

(i) $\bar{a}_{n0} = 1,\ n = 0,1,\ldots,$

(ii) $a_{n-1,v} \geq a_{nv}$ *for* $n \geq v+1$,

(iii) $na_{nn} = O(1),\ 1 = O(na_{nn})$,

(iv) $\sum_{v=0}^{n} a_{vv}\left|\hat{a}_{n,v}\right| = O(a_{nn})$.

(v) $\sum_{n=v+1}^{m+1} n^{k-1}\left|\hat{a}_{n,v+1}\right|^k = O(1)$.

If (X_n) *is a positive monotonic nondecreasing sequence and if* (λ_n) *is a sequence of constants satisfying*

(vi) $\lambda_n = o(1)$,

(vii) $\sum_{n=1}^{m} nX_n\left|\Delta^2\lambda_n\right| = O(1)$,

(viii) $\sum_{n=1}^{m} \dfrac{a_{nn}|t_n|^k}{X_n^{k-1}} = O(X_m)$, *where* $t_n = \dfrac{1}{n+1}\sum_{k=1}^{n} ka_k$,

then the series $\sum a_n\lambda_n$ *is summable* $|A|_k$, $k \geq 1$.

It may be mentioned as well that whenever $X_n \to \infty$, condition (viii) of theorem 2.1 is weaker than condition (vii) of theorem 1.1. For if (viii) of theorem 1.1 is satisfied, then $X_n \to \infty$ implies that $\lambda_n \to 0$, while if $\lambda_n \to 0$, then by choosing

$$\lambda_n = n^{-1/2},\ X_n = n^{\epsilon+(1/2)},\ \epsilon > 0,$$

we obtain $|\lambda_n| X_n = O(n^{\epsilon}) \neq O(1)$.

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Although we add conditions (v), but this condition with (vi) caused for no losing powers via the proof. As an example in the proof of theorem 1.1, $|\lambda_n|^{k-1}$ has been replaced by $O(1)$, and this is not the case in theorem 2.1.

Lemma 2.2. *Condition* (viii) *of theorem* 2.1 *is weaker than* (vii) *of theorem* 1.1

Proof. If (viii) holds, then we have

$$\sum_{n=1}^{m}\frac{a_{nn}|t_n|^k}{X_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right)\sum_{n=1}^{m}a_{nn}|t_n|^k = O(X_m),$$

while if (vii) is satisfied then,

$$\begin{aligned}\sum_{n=1}^{m}a_{nn}|t_n|^k &= \sum_{n=1}^{m}\frac{a_{nn}}{X_n^{k-1}}|t_n|^k X_n^{k-1}\\ &= \sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}\frac{a_{vv}|t_v|^k}{X_v^{k-1}}\right)\Delta X_n^{k-1} + \left(\sum_{n=1}^{m}\frac{a_{nn}|t_n|^k}{X_n^{k-1}}\right)X_m^{k-1}\\ &= O(1)\sum_{n=1}^{m-1}X_n\left|\Delta X_n^{k-1}\right| + O(X_m)X_m^{k-1}\\ &= O(X_{m-1})\sum_{n=1}^{m-1}\left(X_{n+1}^{k-1}-X_n^{k-1}\right) + O\left(X_m^k\right)\\ &= O(X_{m-1})\left(X_m^{k-1}-X_1^{k-1}\right) + O\left(X_m^k\right)\\ &= O\left(X_m^k\right) \neq O(X_m), \text{ for } k>1.\end{aligned}$$

Lemma 2.3. *Conditions* (vi)-(vii) *of theorem 2.1 implies*

$$nX_n|\Delta\lambda_n| = O(1), \qquad (2.1)$$

$$\sum_{n=1}^{\infty}X_n|\Delta\lambda_n| < \infty, \qquad (2.2)$$

and (2.2) *implies*

$$|\lambda_n|X_n = O(1). \qquad (2.3)$$

Proof. Since $\lambda_n \to 0$, then $\Delta\lambda_n \to 0$, and hence

$$nX_n|\Delta\lambda_n| = nX_n\sum_{v=n}^{\infty}\Delta|\Delta\lambda_v| = O(1)\,nX_n\sum_{v=n}^{\infty}\left|\Delta^2\lambda_v\right| = O(1)\sum_{v=n}^{\infty}vX_v\left|\Delta^2\lambda_v\right| = O(1).$$

$$\sum_{n=1}^{m}X_n|\Delta\lambda_n| = \sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}X_v\right)\Delta|\Delta\lambda_n| + \left(\sum_{n=1}^{m}X_n\right)|\Delta\lambda_m|$$

$$= \mathrm{O}(1)\sum_{n=1}^{m-1} nX_n\left|\Delta^2\lambda_n\right| + \mathrm{O}(1)\, mX_m\left|\Delta\lambda_m\right| = \mathrm{O}(1)$$

As $\lambda_n \to 0,$

$$\left|\lambda_n\right| X_n = X_n \sum_{v=n}^{\infty} \Delta\left|\lambda_v\right| = \mathrm{O}(1) \sum_{v=n}^{\infty} X_v\left|\Delta\lambda_v\right| = \mathrm{O}(1).$$

Lemma 2.4. *Under the conditions of theorem 1.2,*

$$\sum_{v=0}^{n-1}\left|\Delta_v \hat{a}_{nv}\right| = \mathrm{O}\left(a_{nn}\right), \tag{2.4}$$

$$\sum_{n=v+1}^{m+1}\left|\Delta_v \hat{a}_{nv}\right| = \mathrm{O}\left(a_{vv}\right). \tag{2.5}$$

$$\sum_{n=v+1}^{m+1}\left|\hat{a}_{n,v+1}\right| = \mathrm{O}(1). \tag{2.6}$$

For the proof see [3].

Proof of Theorem 1.2.

Let Y_n denote the nth term of A-transform of the series $\sum a_n\lambda_n$, then

$$\begin{aligned}
Y_n &:= \sum_{v=0}^{n} \hat{a}_{nv}\lambda_v a_v = \sum_{v=1}^{n} v\, a_v \frac{\hat{a}_{nv}\lambda_v}{v} \\
&= \sum_{v=1}^{n-1}\left(\sum_{r=1}^{v} r\, a_r\right)\Delta_v\left(\frac{\hat{a}_{nv}\lambda_v}{v}\right) + \left(\sum_{v=1}^{n} v\, a_v\right)\left(\frac{a_{nn}\lambda_n}{n}\right) \\
&= \sum_{v=1}^{n-1}(v+1)t_v\left(\frac{1}{v(v+1)}\hat{a}_{nv}\lambda_v + \frac{1}{v+1}\Delta_v\hat{a}_{nv}\lambda_v + \frac{1}{v+1}\hat{a}_{n,v+1}\Delta\lambda_v\right) + \frac{n+1}{n}t_n a_{nn}\lambda_n \\
&= \sum_{v=1}^{n-1}\left(\frac{1}{v}t_v\hat{a}_{nv}\lambda_v + t_v\Delta_v\hat{a}_{nv}\lambda_v + t_v\hat{a}_{n,v+1}\Delta\lambda_v\right) + \frac{n+1}{n}t_n a_{nn}\lambda_n \\
&= Y_{n1} + Y_{n2} + Y_{n3} + Y_{n4}.
\end{aligned}$$

In order to prove the Theorem, by Minkowski's inequality, we have to show that

$$\sum_{n=1}^{\infty} n^{k-1}\left|Y_{nj}\right|^k < \infty, \quad j = 1,2,3,4.$$

By Holder's inequality,

$$\sum_{n=2}^{m} n^{k-1}\left|Y_{n1}\right|^k = \sum_{n=2}^{m} n^{k-1}\left|\sum_{v=1}^{n-1}\frac{1}{v}t_v\hat{a}_{nv}\lambda_v\right|^k$$

$$\leq \sum_{n=2}^{m} n^{k-1} \sum_{v=1}^{n-1} v^{-k} a_{vv}^{1-k} |t_v|^k |\hat{a}_{nv}| |\lambda_v|^k \left(\sum_{v=1}^{n-1} a_{vv} |\hat{a}_{nv}| \right)^{k-1}$$

$$= O(1) \sum_{n=2}^{m} (n a_{nn})^{k-1} \sum_{v=1}^{n-1} v^{-k} a_{vv}^{1-k} |t_v|^k |\hat{a}_{nv}| |\lambda_v|^k$$

$$= O(1) \sum_{v=1}^{m} a_{vv} (v a_{vv})^{-k} |t_v|^k |\lambda_v|^k \sum_{n=v+1}^{\infty} |\hat{a}_{nv}|$$

$$= O(1) \sum_{v=1}^{m} a_{vv} |t_v|^k |\lambda_v|^k$$

$$= O(1) \sum_{v=1}^{m} \frac{a_{vv} |t_v|^k}{X_v^{k-1}} |\lambda_v| (X_v |\lambda_v|)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} \frac{a_{vv} |t_v|^k}{X_v^{k-1}} |\lambda_v|$$

$$= O(1) \sum_{v=1}^{m-1} \Delta |\lambda_v| \sum_{r=1}^{v} \frac{a_{rr} |t_r|^k}{X_r^{k-1}} + O(1) |\lambda_m| \sum_{v=1}^{m} \frac{a_{vv} |t_v|^k}{X_v^{k-1}}$$

$$= O(1) \sum_{v=1}^{m-1} |\Delta \lambda_v| X_v + O|\lambda_m| X_m$$

$$= O(1).$$

in view of (iv), (2.2) ,(2.3) and (2.6) .

$$\sum_{n=2}^{\infty} n^{k-1} |Y_{n2}|^k = \sum_{n=2}^{\infty} n^{k-1} \left| \sum_{v=1}^{n-1} t_v \Delta_v \hat{a}_{nv} \lambda_v \right|^k$$

$$\leq \sum_{n=2}^{\infty} n^{k-1} \sum_{v=1}^{n-1} |t_v|^k |\Delta_v \hat{a}_{nv}| |\lambda_v|^k \left(\sum_{v=1}^{n-1} |\Delta_v \hat{a}_{nv}| \right)^{k-1}$$

$$= O(1) \sum_{n=2}^{\infty} (n a_{nn})^{k-1} \sum_{v=1}^{n-1} |t_v|^k |\Delta_v \hat{a}_{nv}| |\lambda_v|^k$$

$$= O(1) \sum_{v=1}^{\infty} |t_v|^k |\lambda_v|^k \sum_{n=v+1}^{\infty} |\Delta_v \hat{a}_{nv}|$$

$$= O(1) \sum_{v=1}^{\infty} a_{vv} |t_v|^k |\lambda_v|^k$$

$$= O(1),$$

as in the case of Y_{n1} .

$$\sum_{n=2}^{m} n^{k-1} |Y_{n3}|^k = \sum_{n=2}^{m} n^{k-1} \left| \sum_{v=1}^{n-1} t_v \hat{a}_{n,v+1} \Delta \lambda_v \right|^k$$

$$\leq \sum_{n=2}^{m} n^{k-1} \sum_{v=1}^{n-1} |t_v|^k X_v^{1-k} |\hat{a}_{n,v+1}|^k |\Delta\lambda_v| \left(\sum_{v=1}^{n-1} X_v |\Delta\lambda_v| \right)^{k-1}$$
$$= O(1) \sum_{n=2}^{m} n^{k-1} \sum_{v=1}^{n-1} |t_v|^k X_v^{1-k} |\hat{a}_{n,v+1}|^k |\Delta\lambda_v|$$
$$= O(1) \sum_{v=1}^{m} |t_v|^k X_v^{1-k} |\Delta\lambda_v| \sum_{n=v+1}^{m} n^{k-1} |\hat{t}_{n,v+1}|^k$$
$$= O(1) \sum_{v=1}^{m} \frac{a_{vv} |t_v|^k}{X_v^{k-1}} v |\Delta\lambda_v|$$
$$= O(1) \sum_{v=1}^{m-1} \Delta(v|\Delta\lambda_v|) \sum_{r=1}^{v} \frac{a_{rr}|t_r|^k}{X_r^{k-1}} + O(1) m |\Delta\lambda_m| \sum_{v=1}^{m} \frac{a_{vv}|t_v|^k}{X_v^{k-1}}$$
$$= O(1) \sum_{v=1}^{m-1} X_v |\Delta\lambda_v| + O(1) \sum_{v=1}^{m-1} v X_v |\Delta^2 \lambda_v| + O(1) m X_m |\Delta\lambda_m|$$
$$= O(1).$$

in view of (2.1), (2.2), (v), and (viii) .

$$\sum_{n=2}^{\infty} n^{k-1} |Y_{n4}|^k = \sum_{n=2}^{\infty} n^{k-1} \left| \frac{n+1}{n} t_n a_{nn} \lambda_n \right|^k$$
$$= O(1) \sum_{n=1}^{\infty} n^{k-1} |t_n|^k a_{nn}^k |\lambda_n|^k$$
$$= O(1) \sum_{n=1}^{\infty} a_{nn} |t_n|^k |\lambda_n|^k = O(1),$$

as in the case of Y_{n2}

This completes the proof of the Theorem .

Corollary 2.5. *Let*

(i) $np_n = O(P_n)$, $P_n = O(np_n)$,

(ii) $\sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k = O(1/P_v^k)$.

If (X_n) *is a positive nondecreasing sequence and the sequence* (λ_n) *is satisfying conditions* (vi)-(vii) *of theorem* 2.1, and

$$\sum_{n=1}^{m} \frac{p_n |t_n|^k}{P_n X_n^{k-1}} = O(X_m), \qquad (2.7)$$

then the series $\sum a_n \lambda_n$ *is summable* $\left|\bar{N}, p_n\right|_k, \ k \geq 1.$

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