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Research Article

ON INDEX SUMMABILITY OF FOURIER SERIES

B.P. Padhy[1], N.K. Panda[2], *U.K. Misra[3] and Mahendra Misra[4]
[1]*Department of Mathematics, Roland Institute of Technology, Berhampur, Odisha*
[2]*Department of Mathematics, Aeronautics College, Sunabedha, Odisha*
[3]*P.G. Department of Mathematics, Berhampur University, Odisha*
[4]*P.G. Department of Mathematics, N.C. College (Autonomous), Jajpur, Odisha*
**Author for Correspondence*

ABSTRACT
A theorem on index summability factors of Fourier Series has been established.

Key Words: *Summability factors, Fourier series*

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INTRODUCTION

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of positive numbers such that

(1.1) $$P_n = \sum_{v=0}^{n} p_v \to \infty, \ as\ n \to \infty \ (P_{-i} = p_{-i} = 0,\ i \geq 1).$$

The sequence-to-sequence transformation

(1.2) $$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \ .$$

defines the sequence of the (N, p_n) mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$.

The series $\sum a_n$ is said to be summable $\left|N, p_n\right|_k, k \geq 1$, if

(1.3) $$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|t_n - t_{n-1}\right|^k < \infty \ .$$

When $p_n = 1\ for\ all\ n\ and\ k = 1, \left|N, p_n\right|_k$ summability is same as $\left|C,1\right|$ summability.For $k = 1, \left|N, p_n\right|_k$ summability is same as $\left|N, p_n\right|$- summability.

The series $\sum a_n$ is said to be summable $\left|N, p_n; \delta\right|_k, \ k \geq 1, \delta \geq 0$,if

(1.4) $$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \left|t_n - t_{n-1}\right|^k < \infty \ .$$

When $\delta = 0, \alpha = 0, \left|N, p_n; \delta\right|_k$ -summability is the same as $\left|N, p_n\right|_k$ -summability.

The series $\sum a_n$ is said to be summable $\left|N, p_n;\ f\left(\frac{P_n}{p_n}\right)\right|_k, k \geq 1$, if

(1.5) $$\sum_{n=1}^{\infty}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\left|t_n - t_{n-1}\right|^k < \infty .$$

In the case when $f\left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)=\left(\frac{P_n}{p_n}\right)^{\delta}$, $\left|N, p_n; f\left(\frac{P_n}{p_n}\right)\right|_k$ - summability is same as $\left|N, p_n; \delta\right|_k$ summability.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi,\pi)$. Without loss of generality, we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

(1.6) $$f(t) \sim \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t)$$

Known Theorem:

Dealing with $\left|\overline{N}, p_n\right|_k, k \geq 1$, summability factors of Fourier series, Bor [1] proved the following theorem:

Theorem-A:

If $\{\lambda_n\}$ is a non-negative and non-increasing sequence such that $\sum p_n \lambda_n < \infty$, where $\{p_n\}$ is a sequence of positive numbers such that $P_n \to \infty$ *as* $n \to \infty$ and $\sum_{v=1}^{n} P_v A_v(t) = 0(P_n)$. Then the factored Fourier series $\sum A_n(t) P_n \lambda_n$ is summable $\left|\overline{N}, p_n\right|_k, k \geq 1$.

We prove an analogue theorem on $\left|N, p_n\right|_k$ - summability, $k \geq 1$, in the following form:

Main Theorem:

Let $\{p_n\}$ is a sequence of positive numbers as defined in (1.2) such that $P_n = p_1 + p_2 + ... + p_n \to \infty$ as $n \to \infty$ and $\{\lambda_n\}$ is a non-negative, non-increasing sequence such that $\sum p_n \lambda_n < \infty$. If

(3.1) (i). $$\sum_{v=1}^{\infty} P_v A_v(t) = O(P_n)$$

(3.2) (ii). $$\sum_{n=v+1}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\left(\frac{p_{n-v-1}}{P_{n-1}}\right) = 0\left(\frac{p_v}{P_v}\right), \text{ as } m \to \infty$$

and

(3.3) (iii). $$P_{n-v-1}\,\Delta\lambda_v = 0(p_{n-v}\,\lambda_v),$$

then the series $\sum A_n(t) P_n \lambda_n$ is summable $\left|N, p_n; f\left(\frac{P_n}{p_n}\right)\right|_k, k \geq 1$.

Required Lemma:

We need the following Lemma for the proof of our theorem.

Lemma [1]:

If $\{\lambda_n\}$ is a non-negative and non-increasing sequence such that $\sum P_n \lambda_n < \infty$, where $\{p_n\}$ is a sequence of positive numbers such that $P_n \to \infty$ *as* $n \to \infty$ then $P_n \lambda_n = 0(1)$ *as* $n \to \infty$ and $\sum P_n \Delta\lambda_n < \infty$.

Proof of the Theorem:

Let $t_n(x)$ be the *n*-th (N, p_n) mean of the series $\sum_{n=1}^{\infty} A_n(x)\, P_n\, \lambda_n$, then by definition we have

$$\begin{aligned} t_n(x) &= \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} \sum_{r=0}^{v} A_r(x)\, P_r\, \lambda_r \\ &= \frac{1}{P_n} \sum_{r=0}^{n} A_r(x)\, P_r\, \lambda_r \sum_{v=r}^{n} p_{n-v} \\ &= \frac{1}{P_n} \sum_{r=0}^{n} A_r(x)\, P_r\, P_{n-r} \lambda_r \;. \end{aligned}$$

Then

$$\begin{aligned} t_n(x) - t_{n-1}(x) &= \frac{1}{P_n} \sum_{r=0}^{n} P_{n-r}\, P_r\, \lambda_r\, A_r(x) - \frac{1}{P_{n-1}} \sum_{r=0}^{n-1} P_{n-r-1}\, P_r \lambda_r A_r(x) \\ &= \sum_{r=1}^{n} \left(\frac{P_{n-r}}{P_n} - \frac{P_{n-r-1}}{P_{n-1}} \right) P_r\, \lambda_r\, A_r(x) \\ &= \frac{1}{P_n P_{n-1}} \sum_{r=1}^{n} \left(P_{n-r}\, P_{n-1} - P_{n-r-1}\, P_n \right) P_r\, \lambda_r A_r(x) \\ &= \frac{1}{P_n P_{n-1}} \left[\sum_{r=1}^{n-1} \Delta\left\{ \left(P_{n-r}\, P_{n-1} - P_{n-r-1}\, P_n \right) \lambda_r \right\} \left(\sum_{v=1}^{r} P_v\, A_v(x) \right) \right] \text{ with } p_0 = 0\,. \\ &= \frac{1}{P_n P_{n-1}} \left[\sum_{r=1}^{n-1} \left(p_{n-r}\, P_{n-1} - p_{n-r-1}\, P_n \right) \lambda_r P_r \right. \\ &\qquad \left. + \sum_{r=1}^{n-1} \left(p_{n-r-1}\, P_{n-1} - P_{n-r-2}\, p_n \right) P_r\, \Delta\lambda_r \right] \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.} \end{aligned}$$

In order to complete the proof of the theorem, using Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(f\left(\frac{P_n}{p_n} \right) \right)^k \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,i}|^k < \infty, \; for\; i = 1,2,3,4.$$

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Now, we have

$$\sum_{n=2}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,1}\right|^k=\sum_{n=2}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{(P_n)^k}\left(\sum_{v=1}^{n-1}p_{n-v}\,P_v\,\lambda_v\right)^k$$

$$\leq\sum_{n=2}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{P_n}\left(\sum_{v=1}^{n-1}p_{n-v}\,(P_v)^k\,(\lambda_v)^k\right)\left(\frac{1}{P_n}\sum_{v=1}^{n-1}p_{n-v}\right)^{k-1}$$

$$=O(1)\sum_{n=2}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{P_n}\sum_{v=1}^{n-1}p_{n-v}\,P_v\,\lambda_v\,(P_v\,\lambda_v)^{k-1}$$

$$=O(1)\sum_{v=1}^{m}P_v\,\lambda_v\sum_{n=v+1}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\frac{p_{n-v}}{p_n}$$

$$=O(1)\sum_{v=1}^{m}p_v\,\lambda_v\quad\text{(Using 3.2)}$$

$$=O(1),\ as\ m\to\infty.$$

$$\sum_{n=2}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n.2}\right|^k=\sum_{n=2}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{(P_n)^k}\left(\sum_{v=1}^{n-1}p_{n-v-1}\,P_v^k\,\lambda_v\right)^k$$

$$\leq\sum_{n=2}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}p_{n-v-1}\,(P_v\,\lambda_v)^k\right)\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_{n-v-1}\right)^{k-1}$$

$$=\sum_{n=2}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_{n-v-1}\,P_v\,\lambda_v(P_v\,\lambda_v)^{k-1}$$

$$=O(1)\sum_{v=1}^{n-1}P_v\,\lambda_v\sum_{n=v+1}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\left(\frac{p_{n-v-1}}{P_{n-1}}\right),$$

$$=O(1)\sum_{v=1}^{n-1}P_v\,\lambda_v,\ \text{Using (3.2)}$$

$$=O(1),\ as\ m\to\infty.$$

Now,

$$\sum_{n=2}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,3}\right|^k=\sum_{n=2}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{(P_n)^k}\left(\sum_{v=1}^{n-1}p_{n-v-1}\,P_v\,\Delta\lambda_v\right)^{k-1}$$

$$\leq\sum_{n=2}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{P_n}\left(\sum_{v=1}^{n-1}p_{n-v-1}\,(P_v\,\Delta\lambda_v)^k\right)\left(\frac{1}{P_n}\sum_{v=1}^{n-1}p_{n-v-1}\right)^{k-1}$$

$$=O(1)\sum_{n=2}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{P_n}\sum_{v=1}^{n-1}p_{n-v-1}\,P_v\,\Delta\lambda_v(P_v\,\Delta\lambda_v)^{k-1}\text{, by the lemma}$$

$$= O(1)\sum_{v=1}^{m} P_v \quad \Delta\lambda_v \sum_{v=1}^{n-1} \left(f\left(\frac{P_n}{p_n}\right)\right)^k \left(\frac{P_n}{p_n}\right)^{k-1} \left(\frac{p_{n-v-1}}{P_n}\right)$$

$$= O(1)\sum_{v=1}^{m} P_v \quad \Delta\lambda_v \text{ , using (3.2)}$$

$$= O(1) \text{ , } m \to \infty .$$

Finally,

$$\sum_{n=2}^{m+1}\left(f\left(\frac{P_n}{p_n}\right)\right)^k \left(\frac{P_n}{p_n}\right)^{k-1} \left|T_{n,4}\right|^k = \sum_{n=2}^{m+1} \left(f\left(\frac{P_n}{p_n}\right)\right)^k \left(\frac{P_n}{p_n}\right)^{k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left(\sum_{v=1}^{n-1} P_v \quad P_{n-v-2} \ \Delta\lambda_v \right)^k$$

$$\leq O(1)\sum_{n=2}^{m+1} \left(f\left(\frac{P_n}{p_n}\right)\right)^k \left(\frac{P_n}{p_n}\right)^{k-1} \frac{1}{\left(P_{n-1}\right)^k} \left(\sum_{v=1}^{n-1} P_v \quad p_{n-v-1} \ \lambda_v \right)^k \text{ , using (3.3)}$$

$$\leq O(1)\sum_{n=2}^{m+1} \left(f\left(\frac{P_n}{p_n}\right)\right)^k \left(\frac{P_n}{p_n}\right)^{k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_{n-v-1} \left(P_v \ \lambda_v\right)^k \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_{n-v-1} \right)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1} \left(f\left(\frac{P_n}{p_n}\right)\right)^k \left(\frac{P_n}{p_n}\right)^{k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_{n-v-1} P_v \ \lambda_v$$

$$= O(1), \ as \ m \to \infty \text{ , as above}$$

This completes the proof of the theorem.

REFERENCE

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