International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Online International Journal Available at http://www.cibtech.org/jpms.htm
2013 Vol. 3 (4) October-December, pp.62-65/Natarajan
Research Article

MADDOX'S THEOREMS FOR THE NATARAJAN METHOD OF SUMMABILITY

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ABSTRACT
In this paper, entries of sequences and infinite matrices are real or complex numbers. The (M, λ_n) method of summability was introduced by Natarajan in 2013(a) and some of its properties were studied in Natarajan 2012, 2013(a), 2013(b), No date. In this paper, we prove Maddox's theorems for the (M, λ_n) method.
2000 Mathematics Subject Classification: 40, 46.
Key Words: *Regular Matrix, the (N, p_n) Method, the (M, λ_n) Method, Convergence Field*

1. Introduction and Preliminaries

Throughout the present paper, entries of sequences and infinite matrices are real or complex numbers. To make the paper self-contained, we recall the following. Given an infinite matrix $A = (a_{nk})$, n, k = 0, 1, 2, ... and a sequence $x = \{x_k\}$, k = 0, 1, 2, ..., by the A-transform of $x = \{x_k\}$, we mean the sequence $A(x) = \{(Ax)_n\}$,

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0,1,2,...,$$

where we suppose that the series on the right converge. If $\lim_{n\to\infty} (Ax)_n = \ell$, we say that $x = \{x_k\}$ is A-summable or summable A to ℓ. If $\lim_{n\to\infty} (Ax)_n = \ell$, whenever $\lim_{k\to\infty} x_k = m$, we say that A is convergence preserving or conservative. If, further, $\ell = m$ we say that A is regular. The following result gives a characterization of a conservative or regular matrix in terms of its entries (see, for instance, (Peyerimhoff, 1969)).

Theorem 1.1. $A = (a_{nk})$ is conservative if and only if

(i) $\sup_{n\geq 0} \sum_{k=0}^{\infty} |a_{nk}| < \infty$;

(ii) $\lim_{n\to\infty} a_{nk} = \delta_k, \quad k = 0,1,2,...;$

and

(iii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \delta.$

Further, A is regular if and only if (i), (ii), (iii) hold with $\delta_k = 0$, k = 0, 1, 2, ... and $\delta = 1$.

Definition 1.2. The Nörlund method (N, p_n) is defined by the infinite matrix (a_{nk}), where

$$a_{nk} = \begin{cases} \dfrac{p_{n-k}}{P_n}, & k \leq n; \\ 0, & k > n, \end{cases}$$

$$P_n = \sum_{k=0}^{n} p_k \neq 0, \ n = 0, 1, 2, ...$$

The following result is known (see (Peyerimhoff, 1969)).

Theorem 1.3. The (N, p_n) method is regular if and only if

(i) $\sum_{k=0}^{n} |p_k| = O(P_n), \quad n \to \infty;$

and

(ii) $p_n = o(P_n), \quad n \to \infty.$

The (M, λ_n) method was introduced by Natarajan in 2013(a) and some of its nice properties were studied in Natarajan 2012, 2013(a), 2013(b), No date .

Definition 1.4. Let $\{\lambda_n\}$ be a sequence such that $\sum_{n=0}^{\infty} |\lambda_n| < \infty.$ Then the (M, λ_n) method is defined by the infinite matrix (b_{nk}), where

$$b_{nk} = \begin{cases} \lambda_{n-k}, & k \le n; \\ 0, & k > n. \end{cases}$$

Remark 1.5. In this context, we note that the (M, λ_n) method reduces to the well-known Y-method when $\lambda_0 = \lambda_1 = \frac{1}{2}$ and $\lambda_n = 0$, $n \ge 2$.

Theorem 1.6. ((Natarajan, 2013(a)), Theorem 2.3). The (M, λ_n) method is regular if and only if

$$\sum_{n=0}^{\infty} \lambda_n = 1.$$

2. Main Results

In this paper, we will prove some theorems of Maddox for the (M, λ_n) method. Maddox (1977, 1979) proved the following results for the (N, p_n) method.

Theorem 2.1. ((Maddox, 1977), Theorem 1) (N, p) = c if and only if

(i) $p = \{p_n\} \in \ell_1$, i.e., $\sum_{n=0}^{\infty} |p_n| < \infty$;

and

(ii) $p(z) \ne 0$ on $|z| \le 1$,

where $p(z) = \sum_{n=0}^{\infty} p_n z^n$, (N, p) is the convergence field of the (N, p_n) method and c is the Banach space of all convergent sequences.

Theorem 2.2. ((Maddox, 1979), Theorem 5) Let $p = \{p_n\} \in \ell_1$ and $\sum_{n=0}^{\infty} p_n \ne 0.$ Then the following three statements are equivalent:

(i) $(N, p) \subset \ell_\infty$;

(ii) (N, p) = c;

(iii) $p(z) \ne 0$ on $|z| \le 1$,

where ℓ_∞ is the Banach space of all bounded sequences.

We need the following result in the sequel.

Theorem 2.3. Let $p_0 \ne 0$, $\{P_n\}$ be bounded and $\sum_{k=0}^{n} |p_k| = O(P_n)$, $n \to \infty$. Then (M, p) = (N, p), where (M, p) is the convergence field of the (M, p_n) method.

Proof. By hypotheses,

$$\sum_{k=0}^{n} |p_k| \le H\,|P_n|, \quad H > 0$$
$$\le H\,M,$$

where $|P_n| \le M$, n = 0, 1, 2, ..., from which it follows that $\sum_{k=0}^{\infty} |p_k| < \infty$ and consequently, the (M, p_n) method is defined. Also,

$$0 < |p_0| \le \sum_{k=0}^{n} |p_k| \le H\,|P_n|, \quad n = 0,1,2,..., \tag{2.1}$$

which implies that $P_n \ne 0$, n = 0, 1, 2, ... Thus the (N, p_n) method is defined. Let $\sum_{k=0}^{\infty} p_k = P$. Then, using (2.1), we have,

$$0 < |p_0| \le \sum_{k=0}^{\infty} |p_k| \le H\,|P|,$$

which, in turn, implies that $P \ne 0$. Let us denote the convergence fields of the (N, p_n), (M, p_n) methods by (N, p), (M, p) respectively. Since $\lim_{n\to\infty} P_n = P \ne 0$, it now follows that (M, p) = (N, p), completing the proof of the theorem.

Remark 2.4. Under the conditions of Theorem 2.3, $\lim_{n\to\infty} \frac{p_n}{P_n} = \frac{0}{P} = 0$, so that the (N, p_n) method is regular, using Theorem 1.3. We also note that the (M, p_n) method need not be regular, though it is always conservative, in view of Theorem 1.1.

Remark 2.5. The (M, p_n) method is well-defined when $\sum_{k=0}^{\infty} |p_k| < \infty$. So $\{P_n\}$ is always bounded, for,

$$|P_n| \le \sum_{i=0}^{n} |p_i| \le \sum_{i=0}^{\infty} |p_i| < \infty.$$

As a consequence of Theorem 2.1, Theorem 2.2 and Theorem 2.3, we have the following results.

Theorem 2.6. Let $p_0 \ne 0$, $\{P_n\}$ be bounded and $\sum_{k=0}^{n} |p_k| = O(P_n)$, $n \to \infty$. Then (M, p) = c if and only if

$p(z) \ne 0$ on $|z| \le 1$,

where $p(z) = \sum_{n=0}^{\infty} p_n z^n$.

Proof. Let (M, p) = c. Under the conditions of the theorem, (M, p) = (N, p), in view of Theorem 2.3. Since (N, p) = c, using Theorem 2.1, we have, $p(z) \ne 0$ on $|z| \le 1$. Conversely, let $p(z) \ne 0$ on $|z| \le 1$. Under the condition of the theorem, $\sum_{n=0}^{\infty} |p_n| < \infty$ and so the (M, p_n) method is defined. In the course of the proof of Theorem 2.3, we noted that $P_n \ne 0$, n = 0, 1, 2, ... Thus the (N, p_n) method is also defined. Now, using Theorem 2.3, (N, p) = (M, p). Again, using Theorem 2.1, we have, (N, p) = c. It now follows that (M, p) = c, completing the proof of the theorem.

The following result can be proved in a similar fashion.

Theorem 2.7. Let $p_0 \neq 0$, $\{P_n\}$ be bounded and $\sum_{k=0}^{n} |p_k| = O(P_n)$, $n \to \infty$. Then the following three statements are equivalent:

(i) $(M, p) \subset \ell_\infty$;
(ii) $(M, p) = c$;
(iii) $p(z) \neq 0$ on $|z| \leq 1$.

We recall the following result, which is well-known (see (Wilansky, 1964), p. 231).

Theorem 2.8. (Mazur-Orlicz) If a conservative matrix sums a bounded divergent sequence, then it sums an unbounded one.

In the above context, we recall that the (M, p_n) method is always conservative.

In view of Theorem 2.8, we have,

Theorem 2.9. There is no $p = \{p_n\}$ such that $(M, p) = \ell_\infty$.

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