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Research Article

ON PRECONCEPT LATTICES

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ABSTRACT
Formal Concept Analysis (FCA), introduced in 1982 by Rudolph Wille, has undergone prestigious development in view of its many applications and the disciplines, beyond mathematics, in which it is used today. The notion of concept is a key one in FCA, and is generalized by the notions of semiconcept, protoconcept and preconcept. This last class contains all the others and has a lattice structure called preconcept lattice. In this paper, we show that the preconcept lattice has a quotient which is anti-isomorphic to the powerset lattice of the set of attributes of the initial context and that the elements of each class of the quotient structure form a lattice which is isomorphic to a powerset lattice. We also propose an approach for determining the number of preconcepts of a given formal context, and an algorithm for finding them.

Keywords: *Formal Context, Concept, Preconcept, Lattice, Preconcept Lattice*

INTRODUCTION AND PRELIMINARIES

Formal Concept Analysis was introduced by Rudolph Wille (Wille, 2009) to mathematize conceptual data analysis and knowledge processing. Numerous studies and researches have contributed to the enrichment of its field of application. One of these advances was the generalization of the notion of concept by that of preconcepts (Stahl and Wille, 1986). Burgmann and Wille proved a few years later (Burgmann and Wille, 2006) that the set of preconcepts for a given context is a lattice with a defined hierarchical order. In this work, we investigate the structure of the preconcept lattice. We show that this lattice has a quotient which is anti-isomorphic to the powerset lattice of the set of attributes of the initial context and that the elements of each class of the quotient structure form a lattice which is isomorphic to a powerset lattice. This allows us to evaluate the number of preconcepts for a given finite context and finding them.

Formal Concept Analysis (FCA) considers a two-dimensional data set and an incidence relation between them, that is the starting point for FCA and it is called a context. Formally, a (formal) **context** is a triplet (O, A, R) where O represents a non-null set of objects, A represents a non-null set of attributes and R a binary relation between these two sets, which specifies the relation between objects and attributes. When an object o has an attribute a in a formal context (O, A, R), we write oRa or $(o, a) \in R$. The incidence relation R is then seen as a subset of $O \times A$ and can also be seen as an application from $O \times A$ to $\{0,1\}$. Thus, $R(o, a) = 1$ when the object o has the attribute a and $R(o, a) = 0$ otherwise. A formal context can be represented by a binary table, Example 1.1 presents the table of a context with three objects representing three customers, and a set of three attributes representing the drinks that these customers usually order in a restaurant.

Example 1.1. We consider the set $O = \{p_1, p_2, p_3\}$ of customers, who will be represented by 1, 2 and 3 respectively, the set $A = \{beer, juice, water\}$ of drinks usually ordered by these customers in a restaurant, here represented by a, b and c respectively, and R the binary relation given by the table below. (O, A, R) is a formal context.

Table 1. Formal context of Example 1.1

R	**A**	**B**	**c**
1	1	1	0
2	0	1	0
3	1	0	1

Given a formal context (O, A, R), the derivation operators allow to find the set of attributes related to a fixed set of objects and vice versa. These operators are defined between $P(O)$ and $P(A)$ as follows for all $X \in P(O)$ and $Y \in P(A)$:
$X' = \{a \in A : R(o,a) = 1, for\ all\ o \in X\}$ and $Y' = \{o \in O : R(o,a) = 1, for\ all\ a \in Y\}$.
X' is the set of attributes shared by all the objects in X and Y' the set of objects that possesses all attributes in Y.

Definition 1.2. (Wille, 2009) Let $\mathbb{K} = (O, A, R)$ be a formal context. A (formal) **concept** of $\mathbb{K}$ is a pair (X, Y) with $X \subseteq O$ and $Y \subseteq A$ such that $X' = Y$ and $Y' = X$. X is called extent and Y is called intent of (X, Y).
The set of all concepts of a formal context $\mathbb{K}$ is denoted by $B(\mathbb{K})$. This set is ordered by the binary relation $\leq$ defined by: $(X_1, Y_1) \leq (X_2, Y_2) \Leftrightarrow X_1 \subseteq X_2$ ($\Leftrightarrow Y_2 \subseteq Y_1$). It is known that $(B(\mathbb{K}), \leq)$ is a complete lattice, called the concepts lattice of $\mathbb{K}$. The notion of concept has been generalized by several other notions, namely: semiconcept, protoconcept and preconcept.

Definition 1.3. (Luksch and Wille, 1996; Stahl and Wille, 1986; Wille, 2000) Let $\mathbb{K} = (O, A, R)$ be a formal context.

- A **semiconcept** of $\mathbb{K}$ is a pair $(X, Y) \subseteq P(A) \times P(O)$ such that $X = Y'$ or $Y = X'$,
- a **protoconcept** of $\mathbb{K}$ is a pair $(X, Y) \subseteq P(A) \times P(O)$ such that $X'' = Y'$ or $Y'' = X'$,
- a **preconcept** of $\mathbb{K}$ is a pair $(X, Y) \subseteq P(A) \times P(O)$ such that $X \subseteq Y'$($\Leftrightarrow$ Y ⊆ X').

It is easy to see that a concept is a semiconcept, a semiconcept is a protoconcept and a protoconcept is a preconcept. In this study, we focus on the notion of preconcept that generalizes all the others. The set of preconcepts of a formal context $\mathbb{K}$ will be denoted by $P(\mathbb{K})$.

Definition 1.4. (Burgmann and Wille, 2006) Let $\mathbb{K} = (O, A, R)$ be a formal context and $(X, Y), (Z, T) \in P(\mathbb{K})$. Then (X, Y) is called a **sub-preconcept** of (Z, T), if $X \subseteq Z$ and $Y \supseteq T$. When (X, Y) is a sub-preconcept of (Z, T), then (Z, T) is called a **super-preconcept** of (X, Y), and we denote by $(X, Y) \preceq (Z, T)$. The binary relation $\preceq$ is an order relation on $P(\mathbb{K})$. It is known that $(P(\mathbb{K}), \preceq)$ is a lattice called the preconcept lattice of $\mathbb{K}$, as it is recalled below. We denote this lattice by $P(\mathbb{K})$.
Proposition 1.5. (Wille, 2004) For a formal context $\mathbb{K} = (O, A, R)$, the ordered set $P(\mathbb{K})$ is a completely distributive complete lattice, which is isomorphic to the concept lattice of the formal context $\mathbb{V}(\mathbb{K}) = (O \cup A,\ A \cup O,\ R \cup (\neq\ O \times A))$, with the infimum and the supremum define by:
$\bigwedge (X_i, Y_i) = (\bigcap X_i, \bigcup Y_i)$ and $\bigvee (X_i, Y_i) = (\bigcup X_i, \bigcap Y_i)$,
for all $(X_i, Y_i) \in P(\mathbb{K}), i \in I$, where I is an index set.
The above proposition identifies the preconcept lattice with the concept lattice of a formal concept. This allows to find all the preconcepts of a given formal context. Let us illustrate this mechanism. Table 2 shows the context $\mathbb{V}(\mathbb{K})$ derived from the context $\mathbb{K}$ of Table 1 and Table 3 gives the concepts of $\mathbb{V}(\mathbb{K})$ (left) as well as the preconcepts of $\mathbb{K}$ (right). For example, the preconcept **16.** is $(3, ac)$. The preconcept lattice diagram is pictured in Figure 1.

Table 2. Formal context $\mathbb{V}(\mathbb{K})$ derived from the formal context $\mathbb{K}$ of Table 1

$\mathbb{V}(\mathbb{K})$	**1**	**2**	**3**	**a**	**b**	**c**
1	0	1	1	1	1	0
2	1	0	1	0	1	0
3	1	1	0	1	0	1
a	1	1	1	0	1	1
b	1	1	1	1	0	1
c	1	1	1	1	1	0

Table 3. Left: concepts of the context $\mathbb{V}(\mathbb{K})$ of Table 2. Right: preconcepts of $\mathbb{K}$.

$B(\mathbb{V}(\mathbb{K}))$	Extent						Intent					
	1	**2**	**3**	**a**	**b**	**c**	**1**	**2**	**3**	**A**	**b**	**C**
1.	1	1	1	1	1	1	0	0	0	0	0	0
2.	1	1	0	1	1	1	0	0	1	0	0	0
3.	1	1	0	1	0	1	0	0	1	0	1	0
4.	1	0	1	1	1	1	0	1	0	0	0	0
5.	1	0	1	0	1	1	0	1	0	1	0	0
6.	1	0	0	1	1	1	0	1	1	0	0	0
7.	1	0	0	1	0	1	0	1	1	0	1	0
8.	1	0	0	0	1	1	0	1	1	1	0	0
9.	1	0	0	0	0	1	0	1	1	1	1	0
10.	0	1	1	1	1	1	1	0	0	0	0	0
11.	0	1	0	1	1	1	1	0	1	0	0	0
12.	0	1	0	1	0	1	1	0	1	0	1	0
13.	0	0	1	1	1	1	1	1	0	0	0	0
14.	0	0	1	1	1	0	1	1	0	0	0	1
15.	0	0	1	0	1	1	1	1	0	1	0	0
16.	0	0	1	0	1	0	1	1	0	1	0	1
17.	0	0	0	1	1	1	1	1	1	0	0	0
18.	0	0	0	1	1	0	1	1	1	0	0	1
19.	0	0	0	1	0	1	1	1	1	0	1	0
20.	0	0	0	1	0	0	1	1	1	0	1	1
21.	0	0	0	0	1	1	1	1	1	1	0	0
22.	0	0	0	0	1	0	1	1	1	1	0	1
23.	0	0	0	0	0	1	1	1	1	1	1	0
24.	0	0	0	0	0	0	1	1	1	1	1	1

$P(\mathbb{K})$	Extent			Intent		
	1	**2**	**3**	**a**	**b**	**c**
1.	1	1	1	0	0	0
2.	1	1	0	0	0	0
3.	1	1	0	0	1	0
4.	1	0	1	0	0	0
5.	1	0	1	1	0	0
6.	1	0	0	0	0	0
7.	1	0	0	0	1	0
8.	1	0	0	1	0	0
9.	1	0	0	1	1	0
10.	0	1	1	0	0	0
11.	0	1	0	0	0	0
12.	0	1	0	0	1	0
13.	0	0	1	0	0	0
14.	0	0	1	0	0	1
15.	0	0	1	1	0	0
16.	0	0	1	1	0	1
17.	0	0	0	0	0	0
18.	0	0	0	0	0	1
19.	0	0	0	0	1	0
20.	0	0	0	0	1	1
21.	0	0	0	1	0	0
22.	0	0	0	1	0	1
23.	0	0	0	1	1	0
24.	0	0	0	1	1	1

Preconcept lattice

Now, we focus on the structure of preconcept lattice. Given a formal context $\mathbb{K}$, one can find its preconcept lattice structure through $\mathbb{V}(\mathbb{K})$. We show that it is possible to find all the preconcepts of a formal context without needing to compute $B(\mathbb{V}(\mathbb{K}))$.

Given a preconcept $P = (X, Y)$, we set $\pi_1(P) = X$ and $\pi_2(P) = Y$. We have the following proposition:

Proposition 2.1. Let $\mathbb{K} = (O, A, R)$ be a formal context. The binary relation θ defined on $P(\mathbb{K})$ by $P\ \theta\ Q$ iff $\pi_2(P) = \pi_2(Q)$, (i.e, P and Q have the same intent) is a congruence relation.

Proof. To prove that θ is an equivalent relation is straightforward. The proof is completed by the compatibility with the infimum and the supremum shown below:

Let P, Q, U and V be preconcepts of $\mathbb{K}$. Assume that $P\ \theta\ Q$ and $U \theta V$. Then $\pi_2(P) = \pi_2(Q)$ and $\pi_2(U) = \pi_2(V)$. This imply that $\pi_2(P) \cap \pi_2(U) = \pi_2(Q) \cap \pi_2(V)$ and $\pi_2(P) \cup \pi_2(U) = \pi_2(Q) \cup \pi_2(V)$. It follows that $(P \vee U)\ \theta\ (Q \vee V)$ and $(P \wedge U)\ \theta\ (Q \wedge V)$ as required. □

We now look at the quotient of $P(\mathbb{K})$ by the congruence relation θ of Proposition 2.1. On this quotient we define the binary relation $\leqslant$ by:

$$[P] \leqslant [Q] \Leftrightarrow \pi_2(Q) \subseteq \pi_2(P).$$

It is clear that $\leqslant$ is an order relation. We have the following Lemma:

Lemma 2.2. Let $\mathbb{K} = (O, A, R)$ be a formal context, $(P(\mathbb{K})/\theta, \leqslant)$ is a lattice anti-isomorphic to the powerset lattice of A, with the supremum $\veebar$ and the infimum $\barwedge$ are given by
$[P] \barwedge [Q] := [P \wedge Q]$, and $[P] \veebar [Q] := [P \vee Q]$.

Proof. It is clear that $(P(\mathbb{K})/\theta, \leqslant)$ is a lattice
Let consider the map f from $P(\mathbb{K})/\theta$ to $P(A)$ such that $f([P]) = \pi_2(P)$ for all $[P] \in P(\mathbb{K})/\theta$. The map f is clearly well defined and injective. For the surjectivity, if $X \subseteq A$, then (X', X) is a preconcept and $f([(X', X)]) = X$. Hence, f is bijective.
If P and Q are two preconcepts, then

$$f([P] \veebar [Q]) = f([P \vee Q]) = \pi_2(P) \cap \pi_2(Q) = f([P]) \cap f([Q]), \text{ and}$$

$$f([P] \barwedge [Q]) = f([P \wedge Q]) = \pi_2(P) \cup \pi_2(Q) = f([P]) \cup f([Q]),$$ which completes the proof.

On each equivalence class $[P]$, we define the binary relation $\lessdot$ by $\forall P, Q \in [P]$, $P \lessdot Q \Leftrightarrow \pi_1(P) \subseteq \pi_1(Q)$. The relation $\lessdot$ is an order relation that makes $[P]$ a lattice as shown in the following Lemma:

Lemma 2.3. Let $\mathbb{K} = (O, A, R)$ be a formal context and $[Q] \in P(\mathbb{K})/\theta$. Then $([Q], \lessdot)$ is a lattice isomorphic to the powerset lattice of $(\pi_2(Q))'$.

Proof. $([Q], \lessdot)$ is clearly a lattice, with the infimum and the supremum given by:
$(X, \pi_2(Q)) \wedge (Y, \pi_2(Q)) = (X \cap Y, \pi_2(Q))$ and $(X, \pi_2(Q)) \vee (Y, \pi_2(Q)) = (X \cup Y, \pi_2(Q))$
Consider the map h defined from $[Q]$ to $P((\pi_2(Q))')$ as follows: $h(X, \pi_2(Q)) = X$, for all $(X, \pi_2(Q)) \in [Q]$. Clearly, h is well defined and bijective. For $(X, \pi_2(Q)), (Y, \pi_2(Q)) \in [Q]$,

$$h\Big((X, \pi_2(Q)) \vee (Y, \pi_2(Q))\Big) = h\Big((X \cup Y, \pi_2(Q))\Big) = X \cup Y = h\Big((X, \pi_2(Q))\Big) \cup h\Big((Y, \pi_2(Q))\Big)$$

and

$$h((X, \pi_2(Q)) \wedge (Y, \pi_2(Q))) = h((X \cap Y, \pi_2(Q))) = X \cap Y = h((X, \pi_2(Q))) \cap h((Y, \pi_2(Q))),$$

which completes the proof.

Figure 1 illustrates Proposition 2.1. and Lemma 2.3. for Example 1.1. Concepts with the same colour represent equivalence classes through the relation θ of Proposition 2.1. We can see that these classes are lattices in accordance with Lemma 2.3. From Lemmas 2.2. and 2.3., we have the following theorem which evaluates the number of preconcepts of a given finite formal context:

Theorem 2.4. Let $\mathbb{K} = (O, A, R)$ be a formal context and $P(A) = \{X_1, \cdots, X_n\}$ the powerset of A. Then, the number of preconcepts of $\mathbb{K}$ is $N_{\mathbb{K}} = \sum_{i=1}^{n} 2^{|X_i'|}$.

Algorithm 1 below is based on Lemmas 2.2. and 2.3., and finds all the preconcepts of a given formal context. One can easily see that the algorithm has an exponential complexity, w.r.t the size of the input context,
Table 4 below uses Example 1.1. to show how Algorithm 1 works. Each line shows how new preconcepts are found, and updating is done by adding these preconcepts to the set of preconcepts. The last column shows the number of preconcepts already found.

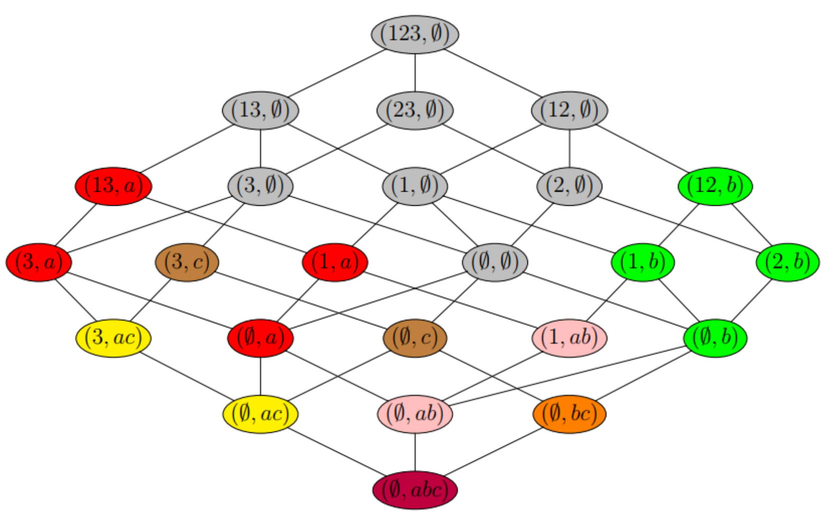

Figure 1. Hasse diagram of the preconcept lattice of the context in Table 1

Algorithm 1: Find preconcepts

Input: a formal context $\mathbb{K} = (O, A, R)$.
Output: the set of preconcepts $P(\mathbb{K})$ of $\mathbb{K}$.
Find $P(A) = \{X_1, X_2, \ldots, X_{2^{|A|}}\}$; // the powerset of A
foreach $i \in \{1, \ldots, 2^{|A|}\}$ **do**
    $C = X_i{'}$;
    find $P(C) = \{Y_1, Y_2, \ldots, Y_{2^{|C|}}\}$;
    foreach $j \in \{1, \ldots, 2^{|C|}\}$ **do**
        Add (C_j, X_i) to $P(\mathbb{K})$;
    end
end

Table 4. Execution of Algorithm 1 on the context of Example 1.1.

$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$					
i	X_i	$C = X_i{'}$	$P(C)$	$P(\mathbb{K})$	$N_{\mathbb{K}}$
1	$\emptyset$	$\{1,2,3\}$	$\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$	$(\emptyset, \emptyset), (\{1\}, \emptyset), (\{2\}, \emptyset), (\{3\}, \emptyset), (\{1,2\}, \emptyset), (\{1,3\}, \emptyset), (\{2,3\}, \emptyset), (\{1,2,3\}, \emptyset)$	8
2	$\{a\}$	$\{1,3\}$	$\emptyset, \{1\}, \{3\}, \{1,3\}$	$(\emptyset, \{a\}), (\{1\}, \{a\}), (\{3\}, \{a\}), (\{1,3\}, \{a\})$	12
3	$\{b\}$	$\{1,2\}$	$\emptyset, \{1\}, \{2\}, \{1,2\}$	$(\emptyset, \{a\}), (\{1\}, \{a\}), (\{2\}, \{a\}), (\{1,2\}, \{a\})$	16
4	$\{c\}$	$\{3\}$	$\emptyset, \{3\}$	$(\emptyset, \{c\}), (\{3\}, \{c\})$	18
5	$\{a,b\}$	$\{1\}$	$\emptyset, \{1\}$	$(\emptyset, \{a,b\}), (\{1\}, \{a,b\})$	20
6	$\{a,c\}$	$\{3\}$	$\emptyset, \{3\}$	$(\emptyset, \{a,c\}), (\{3\}, \{a,c\})$	22
7	$\{b,c\}$	$\emptyset$	$\emptyset$	$(\emptyset, \{b,c\})$	23
8	$\{a,b,c\}$	$\emptyset$	$\emptyset$	$(\emptyset, \{a,b,c\})$	24

CONCLUSION AND FUTURE WORK

In this paper, we have proposed how to determine the number of preconcepts of a given formal context, and through an algorithm how to find all of them. We plan to use this algorithm to construct the Hasse diagram of the preconcept lattice. Knowing that the preconcept lattice contains all concepts, semi-concepts and protoconcepts, we will examine the possibility of identifying these subclasses in the preconcept lattice.

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