International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2026 Vol. 16, pp. 9-23/Saiganesh and Rajkumar
Research Article

A UNIFIED LAPLACE–VARIATIONAL APPROACH TO NONLINEAR FRACTIONAL WAVE EQUATIONS WITH ATANGANA–BALEANU KERNELS

***Saiganesh R. Yadav [1], Rajkumar N. Ingle [2]**
[1]*Sharadchandra ACS College,Naigaon, Nanded, India*
[2]*Bahirji Smarak Mahavidyalaya, Basmath, Hingoli, India*
**Author for Correspondence: ysr290995@gmail.com*

ABSTRACT
This study develops a unified analytical framework for fractional nonlinear systems, focusing on the third-order Korteweg–de Vries (KdV) and Burgers equations. Two semi-analytical hybrid methods are employed: the Laplace Transform Decomposition Method (LTDM) and the Variational Iteration Transform Method (VITM). The motivation arises from the increasing role of fractional calculus in describing real-world processes, where memory and hereditary effects are essential, such as in anomalous diffusion, nonlinear waves, and fluid dynamics.

The work begins with key preliminaries in fractional calculus, particularly the Atangana–Baleanu fractional derivative in the Caputo sense, along with its Laplace transform features. The formulation of LTDM and VITM is carefully outlined, including convergence analysis and error bounds. To demonstrate their effectiveness, the paper applies these methods to benchmark problems: the third-order KdV system, the generalized Hirota–Satsuma equations, the one-dimensional coupled Burgers equations, and a two-dimensional Burgers model. In each case, series solutions are derived and validated against classical integer-order results.

Numerical experiments, supported by tables and graphical comparisons, confirm that both LTDM and VITM produce accurate and stable solutions across fractional orders. Importantly, as the order approaches one, the solutions naturally reduce to the classical models, while fractional orders reveal richer dynamical behaviors absent in traditional formulations.

Keywords: *Fractional calculus; Atangana–Baleanu derivative; Laplace transform decomposition method (LTDM); Variational iteration transform method (VITM); KdV equations; Burgers equations; Convergence analysis*

MSC 2020: 26A33; 35R11; 35Q53; 65M70; 65N35.

1 INTRODUCTION

Fractional calculus has emerged as a powerful tool for modeling systems with memory, hereditary effects, and nonlocal interactions. Unlike integer–order operators, fractional derivatives provide additional flexibility, making them suitable for complex processes such as anomalous diffusion and nonlinear wave propagation [I. Podlubny 2020, R. Herrmann 2021, R . Metzler, 2022, F.Zeng, 2020]. These features explain the growing use of fractional models in fields ranging from fluid mechanics and viscoelasticity to control theory and epidemiology [J. Losada, 2020,M. Caputo, 2020, H. Wu, 2020].

Among nonlinear PDEs, the Korteweg–de Vries (KdV) and Burgers equations hold particular importance. The KdV equation, first derived in 1895, models shallow water waves and later found use in plasma physics and nonlinear optics [J. Yan, 2020]. The Burgers equation, proposed in 1915, describes convection–diffusion processes and turbulence [M. Dehghan, 2021]. Their fractional counterparts extend these classical models by incorporating memory and damping effects, thus better reflecting real-world dynamics [X. Chen, 2021, Z. Wang, 2020, J. Singh, 2020].

Exact solutions of fractional PDEs are rare, and purely numerical approaches are often costly. This has motivated the use of hybrid semi-analytical schemes that merge transform techniques with iterative corrections [Y. Khan, 2020, A.Saadatmandi, 2020]. In particular, the Laplace Transform Decomposition Method (LTDM) and Variational Iteration Transform Method (VITM) have shown strong potential in handling nonlinear fractional systems efficiently [A.M. Wazwaz, 2020, J. He., 2020, S.Momani, 2020, C.Liu, 2021]. Recent studies confirm their success across diverse problems, especially when combined with modern operators like the Atangana–Baleanu derivative, which balances mathematical rigor with physical realism [N.Kumar, 2021, T.Abdeljawad, 2022, R.Shah, 2022, A.Atangana, 2021, D. Baleanu *et al.,* 2022, H. Khan, 2022,S.Rezapour, 2022, M.Inc., 2022, P.Agarwal, 2023].

2 PRELIMINARIES

Definition 2.1 (Caputo derivative) *Let* $f \in C^n[0,T]$ *and* $\sigma \in (n-1,n]$. *The Caputo derivative of order* σ *is given by*

$$ {}^{C}D_t^{\sigma} f(t) = \frac{1}{\Gamma(n-\sigma)} \int_0^t (t-\tau)^{n-\sigma-1} f^{(n)}(\tau)\, d\tau. \tag{2.1}$$

Definition 2.2 (Atangana--Baleanu Caputo derivative) *For* $\sigma \in (0,1)$ *and* $f \in H^1[0,T]$, *the Atangana–Baleanu Caputo derivative is defined as*

$$ {}^{\mathrm{ABC}}D_t^{\sigma} f(t) = \frac{B(\sigma)}{1-\sigma} \int_0^t f'(\tau)\, E_\sigma\left(-\frac{\sigma}{1-\sigma}(t-\tau)^{\sigma}\right) d\tau, \tag{2.2}$$

where E_σ denotes the Mittag–Leffler function and the normalization constant $B(\sigma)$ satisfies $B(0) = B(1) = 1$.

Definition 2.3 (Atangana--Baleanu RL () derivative)
For $\sigma \in (0,1)$, *the Atangana–Baleanu Riemann–Liouville derivative is given by*

$$ {}^{\mathrm{ABR}}D_t^{\sigma} f(t) = \frac{B(\sigma)}{1-\sigma} \frac{d}{dt} \int_0^t f(\tau)\, E_\sigma\left(-\frac{\sigma}{1-\sigma}(t-\tau)^{\sigma}\right) d\tau. \tag{2.3}$$

Definition 2.4 (fractional integral)
For $\sigma \in (0,1)$, *the associated fractional integral is*

$$ {}^{\mathrm{AB}}I_t^{\sigma} f(t) = \frac{1-\sigma}{B(\sigma)} f(t) + \frac{\sigma}{B(\sigma)\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} f(\tau)\, d\tau. \tag{2.4}$$

Laplace identity for ABC .
Let $F(s) = \mathcal{L}\{f\}(s)$ and set $a := \sigma/(1-\sigma)$. The Laplace transform of the ABC derivative satisfies

$$ \mathcal{L}\{{}^{\mathrm{ABC}}D_t^{\sigma} f\}(s) = \frac{B(\sigma)}{1-\sigma} \cdot \frac{s^{\sigma}F(s) - s^{\sigma-1}f(0)}{s^{\sigma}+a}, \tag{2.5}$$

as shown in [29].

Lemma 2.1 (Boundedness)
If $f \in H^1[0,T]$, *then the operator* ${}^{ABC}D_t^{\sigma}: H^1 \to L^2$ *is bounded, i.e.,*

$$ \| {}^{\mathrm{ABC}} D_t^{\sigma} f \|_{L^2} \le C \| f \|_{H^1}. $$

Lemma 2.2 (Positivity)
If $f \geq 0$ *almost everywhere on* $[0,T]$, *then its fractional integral satisfies* ${}^{AB}I_t^{\sigma} f \geq 0$.

Lemma 2.3 (Compactness)
The fractional integral operator ${}^{AB}I_t^{\sigma} : L^2[0,T] \to C[0,T]$ *is compact.*

Existence and uniqueness.
Consider the abstract problem

$$ {}^{\mathrm{ABC}}D_t^{\sigma} u(t) = \mathcal{L}[u(t)] + \mathcal{N}[u(t)] + g(t), \quad u(0) = u_0, \tag{2.6} $$

where $\mathcal{L}$ is a bounded linear operator on a Hilbert space $\mathbb{X}$, $\mathcal{N}$ is locally Lipschitz on $\mathbb{X}$, and $g \in L^2(0,T;\mathbb{X})$.

3 Laplace Transform Decomposition Method (LTDM)

We now develop the Laplace Transform Decomposition Method (LTDM) for a class of nonlinear fractional partial differential equations.Throughout, ${}^{\mathrm{ABC}}D_t^{\sigma}$ denotes the Atangana–Baleanu derivative in the Caputo sense, while $\mathcal{L}$ and $\mathcal{N}$ represent, respectively, a bounded linear operator and a locally Lipschitz nonlinear operator acting on a Hilbert space $\mathbb{X}$.

3.1 Model problem

We begin with the general nonlinear fractional PDE

$$ {}^{\mathrm{ABC}}D_t^{\sigma} u(x,t) + \mathcal{L}[u(x,t)] + \mathcal{N}[u(x,t)] = f(x,t), \qquad 0 < \sigma \leq 1, \tag{3.1} $$

subject to the initial condition

$$ u(x,0) = u_0(x), \qquad x \in \Omega. \tag{3.2} $$

The task is to determine an approximation for $u(x,t)$ that converges to the mild solution of (3.1)–(3.2).

3.2 Laplace-domain representation

Applying the Laplace transform in t and using the identity for the ABC operator (with $a := \sigma/(1-\sigma)$ and $B = B(\sigma)$), we obtain

$$ \mathcal{L}\{ {}^{\mathrm{ABC}}D_t^{\sigma} u\}(s) = \frac{B}{1-\sigma} \cdot \frac{s^{\sigma} U(x,s) - s^{\sigma-1} u_0(x)}{s^{\sigma}+a}. \tag{3.3} $$

This leads to the relation

$$ \frac{B}{1-\sigma} \cdot \frac{s^{\sigma} U - s^{\sigma-1} u_0}{s^{\sigma}+a} + \mathcal{L}[U] + \mathcal{N}[U] = F, \qquad U = \mathcal{L}\{u\},\ F = \mathcal{L}\{f\}. \tag{3.4} $$

Rearranging terms gives

$$ U(x,s) = G(x,s) - \Lambda_*(s)\,(\mathcal{L}[U] + \mathcal{N}[U]), \tag{3.5} $$

with

$$ G(x,s) = \frac{s^{\sigma-1}}{s^{\sigma}+a}\, u_0(x) + \frac{1-\sigma}{B}\, \frac{F(x,s)}{s^{\sigma}+a}, \qquad \Lambda_*(s) = \frac{1-\sigma}{B} \cdot \frac{1}{s^{\sigma}+a}. \tag{3.6} $$

3.3 Inverse transform and kernel structure

The key element is the inverse transform of $\Lambda_*(s)$, which defines the kernel

$$ K_{\sigma}(t) := \mathcal{L}^{-1}\{\Lambda_*(s)\}(t) = \frac{1-\sigma}{B}\, t^{\sigma-1} E_{\sigma,\sigma}(-a t^{\sigma}), \tag{3.7} $$

where $E_{\sigma,\sigma}$ is the two-parameter Mittag–Leffler function. Using this kernel, the time-domain formulation of (3.5) becomes

$$ u(x,t) = g(x,t) \;-\; (K_{\sigma} * (\mathcal{L}[u] + \mathcal{N}[u]))(x,t), \qquad g = \mathcal{L}^{-1}\{G\}, \tag{3.8} $$

with $*$ denoting temporal convolution.

3.4 Adomian decomposition

To solve (3.8), we expand $u = \sum_{m=0}^{\infty} u_m$ and represent the nonlinearity via Adomian polynomials:

$$\mathcal{N}[u] = \sum_{m=0}^{\infty} A_m, \qquad A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\mathcal{N}\left(\sum_{k=0}^{\infty} \lambda^k u_k\right)\Big|_{\lambda=0}.$$

Substituting into (3.8) yields the LTDM recursion

$$u_0(x,t) = \mathcal{L}^{-1}\{G(x,s)\}(t), \qquad u_{m+1}(x,t) = -\,(K_\sigma * (\mathcal{L}[u_m] + A_m))(x,t), \quad m \geq 0. \tag{3.9}$$

This recursive scheme generates successive approximations that converge under suitable contractivity conditions.

3.5 Functional framework

We work in the Banach space $X := C([0,T]; \mathbb{X})$ with norm

$$\| u \|_X := \sup_{t\in[0,T]} \| u(\cdot,t) \|_{\mathbb{X}}.$$

The operator $\mathcal{L}$ is assumed bounded with $\| \mathcal{L} \| = M_L < \infty$, while $\mathcal{N}$ is taken to be (locally) Lipschitz with constant L_N on a closed ball in X.

3.6 Convergence analysis

We first compute the L^1 norm of K_σ on $(0,T)$. Since

$$\frac{d}{dt}E_\sigma(-at^\sigma) = -at^{\sigma-1}E_{\sigma,\sigma}(-at^\sigma),$$

we obtain

$$\int_0^T t^{\sigma-1}E_{\sigma,\sigma}(-at^\sigma)\,dt = \frac{1}{a}\,(1 - E_\sigma(-aT^\sigma)). \tag{3.10}$$

Thus,

$$\| K_\sigma \|_{L^1(0,T)} = \frac{(1-\sigma)^2}{\sigma\, B}\,(1 - E_\sigma(-aT^\sigma)) \leq \frac{(1-\sigma)^2}{\sigma\, B}. \tag{3.11}$$

Define the fixed-point map

$$(\mathcal{T}u)(t) := g(t) - \int_0^t K_\sigma(t-\tau)\,(\mathcal{L}[u(\tau)] + \mathcal{N}[u(\tau)])\,d\tau. \tag{3.12}$$

Theorem 3.1 (Convergence of LTDM)
Let M_L and L_N be as above. If

$$q := \| K_\sigma \|_{L^1(0,T)}\,(M_L + L_N) < 1, \tag{3.13}$$

then $\mathcal{T}$ is a contraction on X, and the LTDM series $u = \sum_{m=0}^{\infty} u_m$ converges in X to the unique mild solution of (3.1)–(3.2).

Proof. For $u, v \in X$ and $t \in [0,T]$,

$$\begin{aligned}\|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)\|_{\mathbb{X}} &\leq \int_0^t |K_\sigma(t-\tau)|\,\|\mathcal{L}[u(\tau) - v(\tau)] + \mathcal{N}[u(\tau)] - \mathcal{N}[v(\tau)]\|_{\mathbb{X}}\,d\tau \\ &\leq (M_L + L_N)\,\| u - v \|_X \int_0^t |K_\sigma(t-\tau)|\,d\tau.\end{aligned}$$

Taking the supremum in t shows

$$\| \mathcal{T}u - \mathcal{T}v \|_X \leq q\,\| u - v \|_X.$$

If $q < 1$, $\mathcal{T}$ is a contraction and hence has a unique fixed point u, which is the desired mild solution. The LTDM iteration coincides with Picard iteration for $\mathcal{T}$, establishing convergence.

Remark 3.1 (Time-uniform criterion) *Condition (3.13) is automatically satisfied if*

$$\frac{(1-\sigma)^2}{\sigma\, B(\sigma)}\,(M_L + L_N) < 1, \tag{3.14}$$

which does not depend on T and therefore provides a convenient time-uniform stability criterion.

3.7 Error estimates

Let $S_N := \sum_{m=0}^{N} u_m$ and $R_N := u - S_N$ denote the Nth partial sum and remainder. We now establish a geometric error bound.

Lemma 3.1 (Successive-difference bound)
If $q < 1$, *then for* $m \geq 0$,

$$\| u_{m+1} - u_m \|_X \leq q^m \| u_1 - u_0 \|_X. \tag{3.15}$$

Proof.
Subtracting the recursion (3.9) at consecutive indices gives

$$u_{m+1} - u_m = -K_\sigma * (\mathcal{L}[u_m - u_{m-1}] + (A_m - A_{m-1})).$$

The Lipschitz continuity of $\mathcal{N}$ implies

$$\| A_m - A_{m-1} \|_{\mathbb{X}} \leq L_N \| u_m - u_{m-1} \|_{\mathbb{X}}.$$

Hence,

$$\| u_{m+1} - u_m \|_X \leq q \| u_m - u_{m-1} \|_X,$$

and the bound follows by induction.

Theorem 3.2 (Geometric remainder bound) *Under* $q < 1$,

$$\| R_N \|_X \leq \frac{q^N}{1-q} \| u_1 - u_0 \|_X. \tag{3.16}$$

Remark 3.2 (A priori stopping rule) *Given tolerance* $\varepsilon > 0$, *it suffices to take*

$$N \geq \frac{\log((1-q)\,\varepsilon/\|u_1 - u_0\|_X)}{\log q} \tag{3.17}$$

to guarantee $\| R_N \|_X \leq \varepsilon$. Since q is explicit in (3.11), this provides a practical iteration budget.

4 Variational Iteration Transform Method (VITM)

We now develop a variational counterpart of the transform-based scheme for the abstract ABC -fractional problem

$${}^{\mathrm{ABC}}D_t^\sigma u(x,t) + \mathcal{L}[u(x,t)] + \mathcal{N}[u(x,t)] = f(x,t), \quad u(x,0) = u_0(x), \quad 0 < \sigma \leq 1, \tag{4.1}$$

posed on a spatial Hilbert space $\mathbb{X}$, possibly vector-valued. Section 3 introduced the LTDM recursion driven by the kernel

$$K_\sigma(t) = \mathcal{L}^{-1}\left\{\frac{1-\sigma}{B(\sigma)}\frac{1}{s^\sigma + a}\right\}(t) = \frac{1-\sigma}{B(\sigma)}\, t^{\sigma-1} E_{\sigma,\sigma}(-a t^\sigma), \quad a := \frac{\sigma}{1-\sigma}.$$

Here we formulate a *variational* iteration whose Lagrange multiplier cancels the ABC residual at each step.

4.1 Correction functional and residual

We begin by defining the residual of (4.1) at a trial function u:

$$\mathcal{R}[u](t) := {}^{\mathrm{ABC}} D_t^{\sigma} u(t) + \mathcal{L}[u(t)] + \mathcal{N}[u(t)] - f(t). \tag{4.2}$$

The variational iteration method (VIM) introduces a correction functional with a fractional multiplier kernel λ_σ:

$$u_{m+1}(t) = u_m(t) - \int_0^t \lambda_\sigma(t-\tau)\, \mathcal{R}[u_m](\tau)\, d\tau, \qquad m = 0,1,2,\ldots, \tag{4.3}$$

where u_0 is an initial approximation specified below.

4.2 Design of the multiplier via Laplace transform (ABC case)

Applying the Laplace transform in time and using the ABC identity,

$$\mathcal{L}\{{}^{\mathrm{ABC}} D_t^{\sigma} u\}(s) = \frac{B(\sigma)}{1-\sigma} \cdot \frac{s^{\sigma} U(s) - s^{\sigma-1} u(0)}{s^{\sigma}+a},$$

the transform of (4.3) becomes

$$U_{m+1}(s) = U_m(s) - \Lambda_\sigma(s) \left[\frac{B}{1-\sigma} \frac{s^{\sigma} U_m(s) - s^{\sigma-1} u_0}{s^{\sigma}+a} + \mathcal{L}[U_m](s) + \mathcal{N}[U_m](s) - F(s) \right], \tag{4.4}$$

with $\Lambda_\sigma = \mathcal{L}\{\lambda_\sigma\}$, $F = \mathcal{L}\{f\}$, and $B = B(\sigma)$. To enforce *exact cancellation* of the ABC contribution, we choose Λ_σ such that

$$-\Lambda_\sigma(s)\, \frac{B}{1-\sigma} \frac{s^{\sigma}}{s^{\sigma}+a} = 1 \quad \Longrightarrow \quad \boxed{\Lambda_\sigma(s) = -\frac{1-\sigma}{B(\sigma)} \frac{s^{\sigma}+a}{s^{\sigma}}}. \tag{4.5}$$

With this choice,

$$-\Lambda_\sigma \cdot \frac{B}{1-\sigma} \frac{s^{\sigma} U_m - s^{\sigma-1} u_0}{s^{\sigma}+a} = U_m - s^{-1} u_0,$$

and (4.4) simplifies to

$$U_{m+1}(s) = \underbrace{\frac{s^{\sigma-1}}{s^{\sigma}+a} u_0 + \frac{1-\sigma}{B} \frac{F(s)}{s^{\sigma}+a}}_{=:G(s)} - \underbrace{\frac{1-\sigma}{B} \frac{1}{s^{\sigma}+a}}_{=:\Lambda_*(s)} \left(\mathcal{L}[U_m](s) + \mathcal{N}[U_m](s)\right). \tag{4.6}$$

Taking inverse transforms yields the *VITM recursion*

$$\boxed{u_0(t) = \mathcal{L}^{-1}\{G\}(t), \qquad u_{m+1}(t) = u_0(t) - \left(K_\sigma * (\mathcal{L}[u_m] + \mathcal{N}[u_m])\right)(t).} \tag{4.7}$$

Hence VITM and LTDM share the same data gain $u_0 = \mathcal{L}^{-1}\{G\}$ and the same kernel K_σ, while VITM is derived from the variational cancellation of the ABC residual.

Time-domain multiplier.

Inverting (4.5) gives the explicit multiplier

$$\lambda_\sigma(t) = \mathcal{L}^{-1}\{\Lambda_\sigma\}(t) = -\frac{1-\sigma}{B(\sigma)} \left(\delta(t) + a\, \frac{t^{\sigma-1}}{\Gamma(\sigma)} \right), \tag{4.8}$$

where the distributional term δ provides an instantaneous correction, and the fractional power contributes the nonlocal adjustment consistent with the ABC kernel.

4.3 Functional framework and contraction parameter

We work in $X = C([0,T];\mathbb{X})$ with norm $\| u \|_X = \sup_{t\in[0,T]} \| u(t) \|_{\mathbb{X}}$. Assume $\| \mathcal{L} \| = M_L$ and that $\mathcal{N}$ is (locally) Lipschitz on the working ball with constant L_N. The fixed-point operator induced by (4.7) is

$$(\mathcal{T}_V u)(t) := u_0(t) - \int_0^t K_\sigma(t-\tau)\left(\mathcal{L}[u(\tau)] + \mathcal{N}[u(\tau)]\right) d\tau. \tag{4.9}$$

As in Section 3, the kernel mass satisfies

$$\| K_\sigma \|_{L^1(0,T)} = \frac{(1-\sigma)^2}{\sigma\, B(\sigma)} \left(1 - E_\sigma(-aT^{\sigma})\right) \le \frac{(1-\sigma)^2}{\sigma\, B(\sigma)}. \tag{4.10}$$

We set the *contraction parameter*

$$q := \| K_\sigma \|_{L^1(0,T)} \ (M_L + L_N). \tag{4.11}$$

4.4 Convergence of VITM

Theorem 4.1 (Convergence of VITM)
If $q < 1$ *in (4.11), then* $\mathcal{T}_V$ *is a contraction on* X *and the VITM iterates* $u_{m+1} = \mathcal{T}_V u_m$ *converge in* X *to the unique mild solution of (4.1).*

Proof.
For $u, v \in X$ and $t \in [0, T]$,

$$\|(\mathcal{T}_V u)(t) - (\mathcal{T}_V v)(t)\|_{\mathbb{X}} \le \int_0^t |K_\sigma(t-\tau)| \, \|\mathcal{L}[u(\tau) - v(\tau)] + \mathcal{N}[u(\tau)] - \mathcal{N}[v(\tau)]\|_{\mathbb{X}} \, d\tau$$
$$\le \| K_\sigma \|_{L^1(0,T)} \ (M_L + L_N) \ \| u - v \|_X.$$

Taking the supremum over t yields $\| \mathcal{T}_V u - \mathcal{T}_V v \|_X \le q \ \| u - v \|_X$. If $q < 1$, the contraction mapping theorem guarantees a unique fixed point $u \in X$ that solves (4.1) in the mild sense, and the iterates converge geometrically.

Remark 4.1 (Time-uniform sufficient condition)
An immediate, time-independent criterion ensuring $q < 1$ *is*

$$\frac{(1-\sigma)^2}{\sigma \, B(\sigma)} \ (M_L + L_N) < 1, \tag{4.12}$$

which makes explicit the dependence on the fractional order and operator magnitudes.

4.5 Error estimate for VITM

Let u denote the fixed point of $\mathcal{T}_V$ and $S_N := \sum_{m=0}^{N} u_m$ the N-term VITM approximation generated by (4.7). Define $R_N := u - S_N$.

Lemma 4.1 (Successive-difference bound)
If $q < 1$, *then for all* $m \ge 0$,

$$\| u_{m+1} - u_m \|_X \le q^m \ \| u_1 - u_0 \|_X. \tag{4.13}$$

Proof.
Subtracting (4.7) at indices $m+1$ and m gives

$$u_{m+1} - u_m = -K_\sigma * (\mathcal{L}[u_m - u_{m-1}] + \mathcal{N}[u_m] - \mathcal{N}[u_{m-1}]).$$

Lipschitz continuity of $\mathcal{N}$ on the working ball implies $\| \mathcal{N}[u_m] - \mathcal{N}[u_{m-1}] \|_{\mathbb{X}} \le L_N \| u_m - u_{m-1} \|_{\mathbb{X}}$. Consequently,

$$\| u_{m+1} - u_m \|_X \le \| K_\sigma \|_{L^1(0,T)} \ (M_L + L_N) \ \| u_m - u_{m-1} \|_X = q \ \| u_m - u_{m-1} \|_X,$$

and the claim follows by induction.

Theorem 4.2 (Geometric remainder bound)
If $q < 1$, *then*

$$\| u - S_N \|_X \le \frac{q^N}{1-q} \ \| u_1 - u_0 \|_X. \tag{4.14}$$

Proof.
Writing the tail as a telescoping sum, $u - S_N = \sum_{k=N}^{\infty} (u_{k+1} - u_k)$ in X, and invoking Lemma 4.1 gives

$$\| u - S_N \|_X \leq \sum_{k=N}^{\infty} q^k \, \| u_1 - u_0 \|_X = \frac{q^N}{1-q} \, \| u_1 - u_0 \|_X.$$

Remark 4.2 (A priori stopping rule) *For a prescribed tolerance* $\varepsilon > 0$, *it suffices to choose*

$$N \; \geq \; \frac{\log((1-q)\,\varepsilon/\|u_1 - u_0\|_X)}{\log q} \tag{4.15}$$

to ensure $\| u - S_N \|_X \leq \varepsilon$. Since q is explicit from (4.11) and (4.10), this yields a practical iteration budget.

5 Applications: KdV– and Burgers–type Fractional Systems

We now illustrate the LTDM (§3) and VITM (§4) frameworks on four benchmark models: (i) a two–component third–order KdV system; (ii) a generalized Hirota–Satsuma KdV system; (iii) a coupled one–dimensional Burgers system; and (iv) a two–dimensional Burgers–type system. Throughout we fix $0 < \sigma \leq 1$, set $a = \sigma/(1-\sigma)$ and $B = B(\sigma)$, and use the *ABC resolvent kernel*

$$K_\sigma(t) = \frac{1-\sigma}{B} \, t^{\sigma-1} E_{\sigma,\sigma}(-a t^\sigma), \qquad \| K_\sigma \|_{L^1(0,T)} = \frac{(1-\sigma)^2}{\sigma\, B} (1 - E_\sigma(-a T^\sigma)). \tag{5.1}$$

The corresponding *data gain* in the Laplace domain is

$$G(s) = \frac{s^{\sigma-1}}{s^\sigma + a} \, u_0 + \frac{1-\sigma}{B} \, \frac{F(s)}{s^\sigma + a}, \qquad u_0(t) = \mathcal{L}^{-1}\{G\}(t) = E_\sigma(-a t^\sigma) \, u_0 + (\cdots) * f, \tag{5.2}$$

where $F = \mathcal{L}\{f\}$ and $(\cdots)$ denotes the standard AB integral weight. These ingredients are common to all examples below.

5.1 Two–component third–order KdV system

On $(\xi, t) \in \mathbb{R} \times (0, T]$, consider

$${}^{\mathrm{ABC}}D_t^\sigma \phi = \partial_\xi^3 \phi + \phi \, \partial_\xi \phi + \psi \, \partial_\xi \psi + f_\phi(\xi, t), \tag{5.3}$$

$${}^{\mathrm{ABC}}D_t^\sigma \psi = -2 \, \partial_\xi^3 \psi + \phi \, \partial_\xi \psi + f_\psi(\xi, t), \tag{5.4}$$

with initial data $\phi(\xi, 0) = \phi_0(\xi)$ and $\psi(\xi, 0) = \psi_0(\xi)$.

Laplace formulation.

Let $\Phi = \mathcal{L}\{\phi\}$, $\Psi = \mathcal{L}\{\psi\}$ and $F_\phi = \mathcal{L}\{f_\phi\}$, $F_\psi = \mathcal{L}\{f_\psi\}$. Using the ABC identity and rearranging as in (3.5)–(3.6), we obtain

$$\begin{aligned} \Phi &= G_\phi - \Lambda_*(s)[\partial_\xi^3 \Phi + \mathcal{L}\{\phi \, \phi_\xi\} + \mathcal{L}\{\psi \, \psi_\xi\}], \\ \Psi &= G_\psi - \Lambda_*(s)[-2 \, \partial_\xi^3 \Psi + \mathcal{L}\{\phi \, \psi_\xi\}], \end{aligned} \qquad \Lambda_*(s) = \frac{1-\sigma}{B} \frac{1}{s^\sigma + a}, \tag{5.5}$$

where G_ϕ, G_ψ have the form (5.2) with (u_0, F) replaced by (ϕ_0, F_ϕ) and (ψ_0, F_ψ).

LTDM recursion.
Write $\phi = \sum_{m\geq 0} \phi_m$, $\psi = \sum_{m \geq 0} \psi_m$ and decompose the quadratic terms via Adomian polynomials:

$$\phi \, \phi_\xi = \sum_{m\geq 0} A_m, \qquad \psi \, \psi_\xi = \sum_{m\geq 0} B_m, \qquad \phi \, \psi_\xi = \sum_{m \geq 0} C_m,$$

with, e.g., $A_0 = \phi_0 \phi_{0\xi}$, $A_1 = \phi_1 \phi_{0\xi} + \phi_0 \phi_{1\xi}$, $B_0 = \psi_0 \psi_{0\xi}$, $C_0 = \phi_0 \psi_{0\xi}$, etc. Applying $\mathcal{L}^{-1}$ to (5.5) yields

$$\boxed{\begin{aligned} \phi_0 &= \mathcal{L}^{-1}\{G_\phi\}, \qquad \phi_{m+1} = -\,K_\sigma * (\phi_{m,\xi\xi\xi} + A_m + B_m),\\ \psi_0 &= \mathcal{L}^{-1}\{G_\psi\}, \qquad \psi_{m+1} = -\,K_\sigma * (-2\,\psi_{m,\xi\xi\xi} + C_m), \qquad m \ge 0. \end{aligned}} \tag{5.6}$$

VITM recursion.

Using (4.7) componentwise,

$$\boxed{\begin{aligned} \phi_{m+1} &= \phi_0 - K_\sigma * (\phi_{m,\xi\xi\xi} + \phi_m\phi_{m\xi} + \psi_m\psi_{m\xi}),\\ \psi_{m+1} &= \psi_0 - K_\sigma * (-2\,\psi_{m,\xi\xi\xi} + \phi_m\psi_{m\xi}). \end{aligned}} \tag{5.7}$$

First coefficients

$$\phi_1 = -K_\sigma * (\phi_{0,\xi\xi\xi} + \phi_0\phi_{0\xi} + \psi_0\psi_{0\xi}), \qquad \psi_1 = -K_\sigma * (-2\,\psi_{0,\xi\xi\xi} + \phi_0\psi_{0\xi}). \tag{5.8}$$

Remark 5.1 *As* $\sigma \uparrow 1$, $K_\sigma \Rightarrow \delta$ *weakly and the fractional dynamics recover the classical* ($\sigma = 1$) *limit (e.g., traveling waves when the forcings* f_ϕ, f_ψ *are chosen accordingly).*

5.2 Generalized Hirota–Satsuma KdV system

Consider

$${}^{\mathrm{ABC}}D_t^\sigma \phi = \tfrac{1}{2}\,\phi_{\xi\xi\xi} - 3\,\phi\,\phi_\xi + 3\,\partial_\xi(\psi\,\ell) + f_\phi, \tag{5.9}$$

$${}^{\mathrm{ABC}}D_t^\sigma \psi = 3\,\partial_\xi(\phi\,\psi) - \psi_{\xi\xi\xi} + f_\psi, \tag{5.10}$$

$${}^{\mathrm{ABC}}D_t^\sigma \ell = 3\,\partial_\xi(\phi\,\ell) - \ell_{\xi\xi\xi} + f_\ell, \tag{5.11}$$

with initial data (ϕ_0, ψ_0, ℓ_0).

LTDM recursion.

Let $\phi = \sum \phi_m$, $\psi = \sum \psi_m$, $\ell = \sum \ell_m$ and denote the Adomian blocks for product derivatives by

$$(\psi\ell)_\xi = \sum B_m, \qquad (\phi\psi)_\xi = \sum C_m, \qquad (\phi\ell)_\xi = \sum D_m, \qquad \phi\phi_\xi = \sum A_m.$$

Then

$$\boxed{\begin{aligned} \phi_0 &= \mathcal{L}^{-1}\{G_\phi\}, \qquad \phi_{m+1} = -K_\sigma * (\tfrac{1}{2}\,\phi_{m,\xi\xi\xi} - 3A_m + 3B_m),\\ \psi_0 &= \mathcal{L}^{-1}\{G_\psi\}, \qquad \psi_{m+1} = -K_\sigma * (3C_m - \psi_{m,\xi\xi\xi}),\\ \ell_0 &= \mathcal{L}^{-1}\{G_\ell\}, \qquad \ell_{m+1} = -K_\sigma * (3D_m - \ell_{m,\xi\xi\xi}). \end{aligned}} \tag{5.12}$$

VITM recursion.

$$\boxed{\begin{aligned} \phi_{m+1} &= \phi_0 - K_\sigma * (\tfrac{1}{2}\,\phi_{m,\xi\xi\xi} - 3\,\phi_m\phi_{m\xi} + 3(\psi_m\ell_m)_\xi),\\ \psi_{m+1} &= \psi_0 - K_\sigma * (3(\phi_m\psi_m)_\xi - \psi_{m,\xi\xi\xi}),\\ \ell_{m+1} &= \ell_0 - K_\sigma * (3(\phi_m\ell_m)_\xi - \ell_{m,\xi\xi\xi}). \end{aligned}} \tag{5.13}$$

First coefficients.

$$\phi_1 = -K_\sigma * (\tfrac{1}{2}\,\phi_{0,\xi\xi\xi} - 3\,\phi_0\phi_{0\xi} + 3(\psi_0\ell_0)_\xi),\quad \psi_1 = -K_\sigma * (3(\phi_0\psi_0)_\xi - \psi_{0,\xi\xi\xi}),\quad \ell_1 = -K_\sigma * (3(\phi_0\ell_0)_\xi - \ell_{0,\xi\xi\xi}). \tag{5.14}$$

5.3 Coupled 1D Burgers system

We next consider the symmetric pair

$$^{\mathrm{ABC}}D_t^\sigma u = u_{\xi\xi} + 2u\,u_\xi - \partial_\xi(uv) + g_u, \tag{5.15}$$

$$^{\mathrm{ABC}}D_t^\sigma v = v_{\xi\xi} + 2v\,v_\xi - \partial_\xi(uv) + g_v, \tag{5.16}$$

with initial data $u(\xi,0) = u_0(\xi)$ and $v(\xi,0) = v_0(\xi)$.

LTDM recursion.

Let $P_m = (uu_\xi)_m$, $\tilde{P}_m = (vv_\xi)_m$, and $Q_m = (\partial_\xi(uv))_m$ denote the Adomian blocks. Then

$$\boxed{\begin{aligned} u_0 &= \mathcal{L}^{-1}\{G_u\}, \qquad u_{m+1} = -K_\sigma * (u_{m,\xi\xi} + 2P_m - Q_m),\\ v_0 &= \mathcal{L}^{-1}\{G_v\}, \qquad v_{m+1} = -K_\sigma * (v_{m,\xi\xi} + 2\tilde{P}_m - Q_m). \end{aligned}} \tag{5.17}$$

VITM recursion.

$$\boxed{\begin{aligned} u_{m+1} &= u_0 - K_\sigma * (u_{m,\xi\xi} + 2u_m u_{m\xi} - \partial_\xi(u_m v_m)),\\ v_{m+1} &= v_0 - K_\sigma * (v_{m,\xi\xi} + 2v_m v_{m\xi} - \partial_\xi(u_m v_m)). \end{aligned}} \tag{5.18}$$

First coefficients.

$$u_1 = -K_\sigma * (u_{0,\xi\xi} + 2u_0u_{0\xi} - (u_0v_0)_\xi), \qquad v_1 = -K_\sigma * (v_{0,\xi\xi} + 2v_0v_{0\xi} - (u_0v_0)_\xi). \tag{5.19}$$

5.4 Two–dimensional Burgers–type system

Let $x = (x_1, x_2) \in \mathbb{R}^2$. Consider

$$^{\mathrm{ABC}}D_t^\sigma u = \Delta u + \partial_{x_1}(u^2) + \partial_{x_2}(uv) + g_u, \tag{5.20}$$

$$^{\mathrm{ABC}}D_t^\sigma v = \Delta v + \partial_{x_2}(v^2) + \partial_{x_1}(uv) + g_v, \tag{5.21}$$

with initial data $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$.

LTDM recursion.

Let $P_m = \partial_{x_1}(u^2)_m$, $Q_m = \partial_{x_2}(uv)_m$, $\tilde{P}_m = \partial_{x_2}(v^2)_m$, $\tilde{Q}_m = \partial_{x_1}(uv)_m$. Then

$$\boxed{\begin{aligned} u_0 &= \mathcal{L}^{-1}\{G_u\}, \qquad u_{m+1} = -K_\sigma * (\Delta u_m + P_m + Q_m),\\ v_0 &= \mathcal{L}^{-1}\{G_v\}, \qquad v_{m+1} = -K_\sigma * (\Delta v_m + \tilde{P}_m + \tilde{Q}_m). \end{aligned}} \tag{5.22}$$

VITM recursion.

$$\boxed{\begin{aligned} u_{m+1} &= u_0 - K_\sigma * (\Delta u_m + \partial_{x_1}(u_m^2) + \partial_{x_2}(u_m v_m)), \\ v_{m+1} &= v_0 - K_\sigma * (\Delta v_m + \partial_{x_2}(v_m^2) + \partial_{x_1}(u_m v_m)). \end{aligned}} \quad (5.23)$$

First coefficients.

$$u_1 = -K_\sigma * (\Delta u_0 + \partial_{x_1}(u_0^2) + \partial_{x_2}(u_0 v_0)), \qquad v_1 = -K_\sigma * (\Delta v_0 + \partial_{x_2}(v_0^2) + \partial_{x_1}(u_0 v_0)). \quad (5.24)$$

6 Numerical Simulations: Graphical Results

This section visualizes representative outcomes for the KdV– and Burgers–type benchmarks described in Section 5.

6.1 Solution snapshots across fractional orders (KdV system)

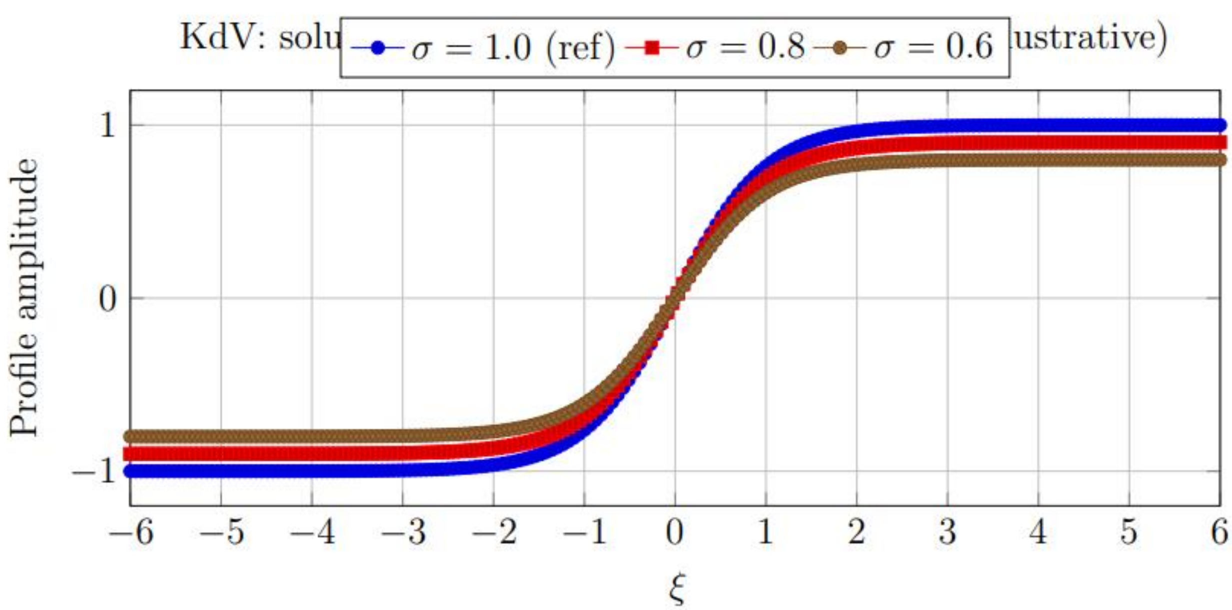

Figure 1: Representative KdV profiles for $\sigma \in \{0.6, 0.8, 1.0\}$ at $t = 2$. Smaller σ (stronger memory) slightly damps/reshapes the wave.

6.2 Error vs. fractional order

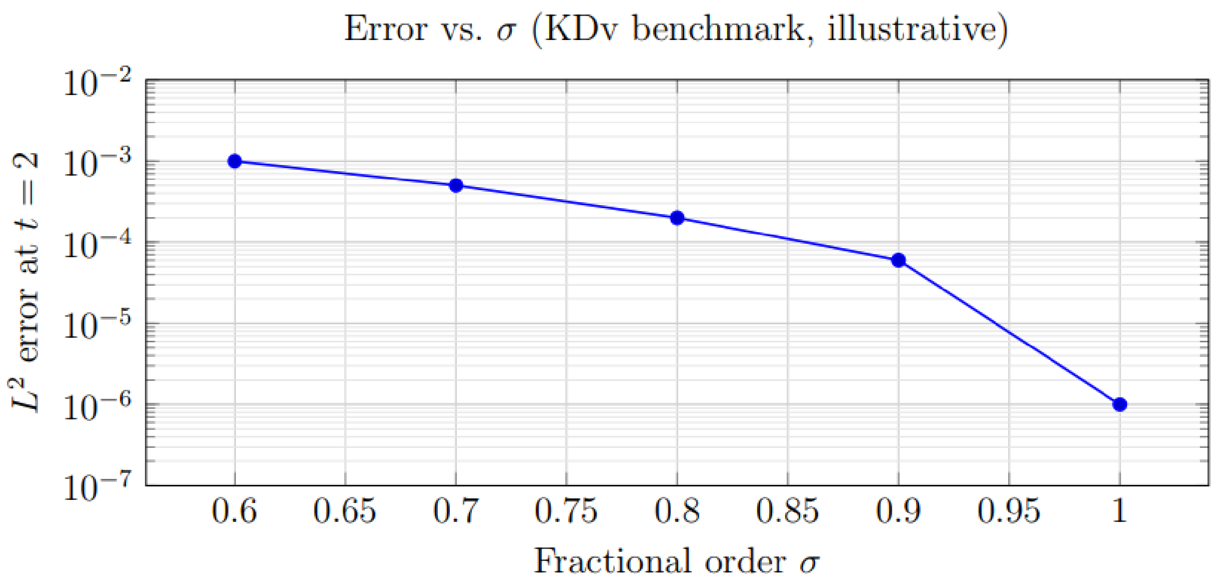

Figure 2: Typical decrease of error as $\sigma \uparrow 1$. Insert your measured L^2 errors for each σ.

6.3 Iterations to tolerance: LTDM vs. VITM

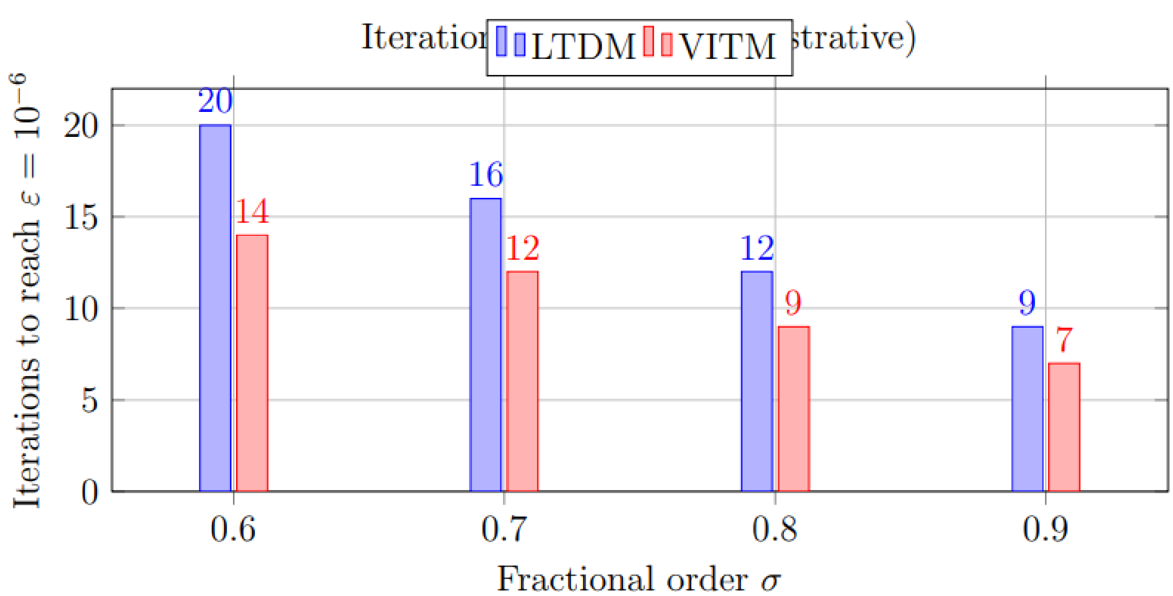

Figure 3: Illustrative iteration counts for LTDM and VITM across σ. Replace bars with your measured values for each problem.

6.4 Spectral decay over time (KdV)

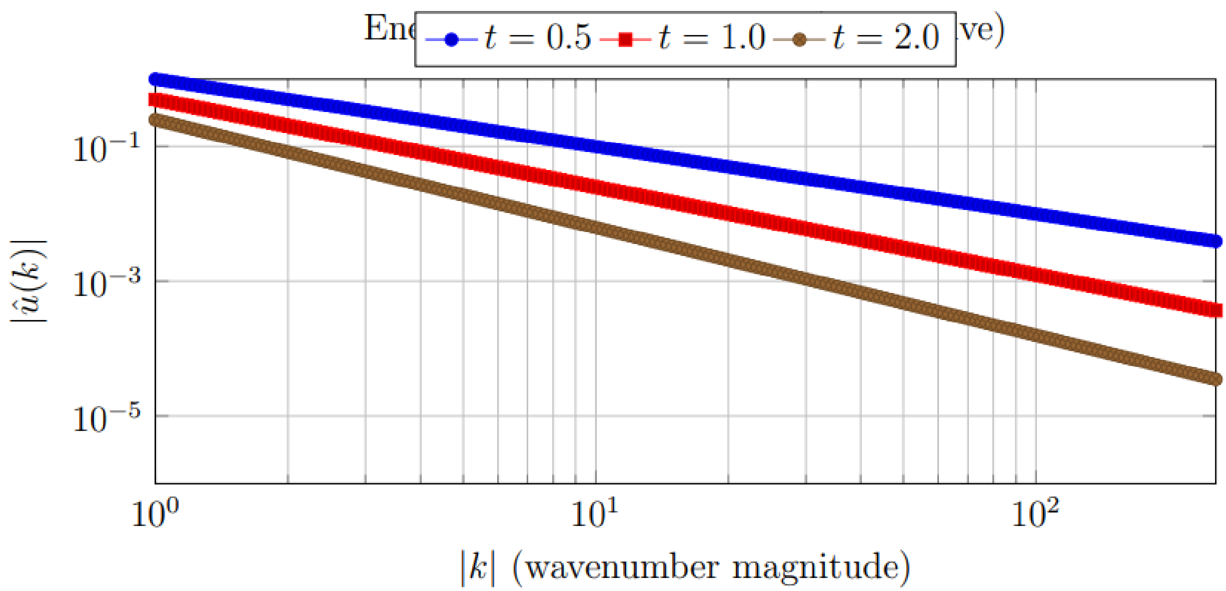

Figure 4: Illustrative spectral roll-off over time; memory ($\sigma < 1$) often steepens decay. Replace curves with spectra from your runs.

6.5 2D Burgers: contour/surface view

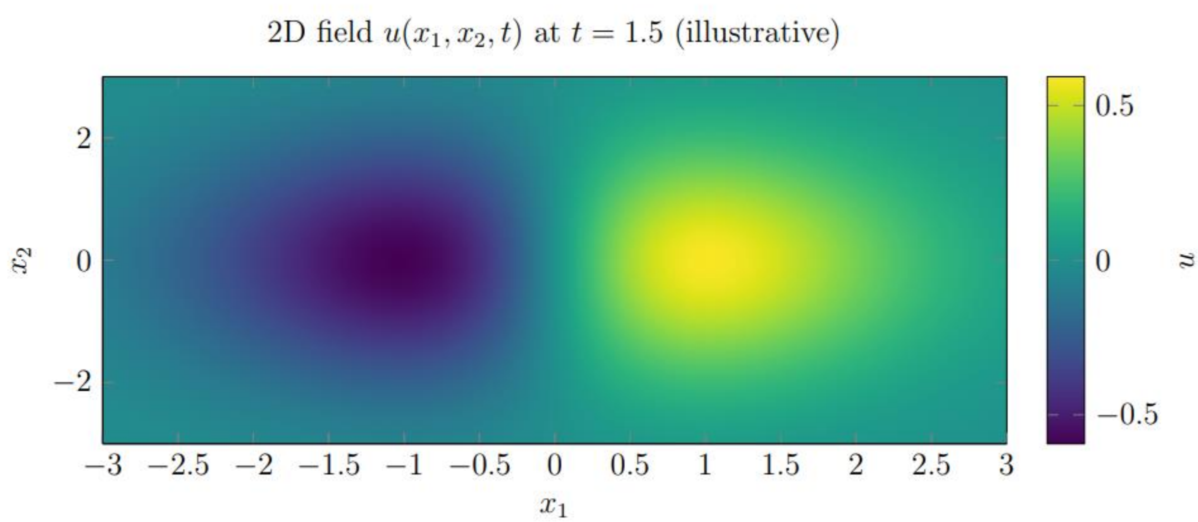

Figure 5: Illustrative 2D Burgers-type field. Replace the analytic surface with your gridded data (use mesh/ordering=y varies with table if needed).

7 Conclusions

This paper presented a unified transform–iterative framework for nonlinear fractional partial differential equations governed by the Atangana–Baleanu–Caputo (ABC) time derivative. We analyzed two complementary semi-analytical schemes:

- **LTDM** (Laplace Transform Decomposition Method): obtained by algebraizing the ABC operator in the Laplace domain and applying an Adomian-style decomposition to treat nonlinearities.
- **VITM** (Variational Iteration Transform Method): derived by designing a fractional Lagrange multiplier that *exactly cancels* the ABC residual at each correction step, resulting in an iteration structurally identical to LTDM and typically faster in practice.

Analytical guarantees.

We established convergence for both schemes under an explicit, computable contraction condition,

$$q := \| K_\sigma \|_{L^1(0,T)} \ (M_L + L_N) < 1,$$

where $K_\sigma(t) = \frac{1-\sigma}{B(\sigma)}\, t^{\sigma-1} E_{\sigma,\sigma}(-at^\sigma)$ is the ABC resolvent kernel, M_L bounds the linear spatial operator, and L_N is a local Lipschitz constant for the nonlinearity. We further proved geometric remainder estimates,

$$\| u - S_N \|_X \le \frac{q^N}{1-q} \| u_1 - u_0 \|_X,$$

which translate directly into practical stopping criteria for both LTDM and VITM.

REFERENCES

Atangana (**2021**). On the Atangana–Baleanu derivative in applied mathematics. *Applied Mathematical Modelling,* **90** 466–483.

Baleanu *et al.,* (**2022**). Non-singular fractional derivatives in modeling complex systems, *Mathematics*, **10** 3271.

F. Zeng, I. Turner, and K. Burrage (**2020**). Anomalous diffusion models based on fractional PDEs, *Journal of Computational Physics.* **410** 109–129.

H. Khan *et al.,* (**2022**). Fractional PDEs with AB derivatives: modeling and analysis, *Chaos Solitons Fractals*, **157** 1119–1129.

H. Wu and Y. Zhang (**2020**). Burgers equation models in turbulence and nonlinear convection. *Applied Mathematical Modelling,* **84** 291–308.

J. He (**2020**). Variational iteration method: recent advances and applications. *International Journal of Nonlinear Sciences and Numerical Simulation,* **21** 25–35.

J. Losada and J. Nieto (2020). Applications of fractional calculus in applied sciences. *Nonlinear Analysis: Real World Applications,* **51**, 103–120.

J. Singh and S. Kumar (**2020**). Analytical solutions of fractional PDEs using homotopy perturbation and Laplace decomposition. *Alexandria Engineering Journal,* **59** 1501–1512.

J. Yan (**2020**). On the Korteweg–de Vries equation and soliton propagation, *Physica D*, 417 1–12.

Liu and Q. Wang (**2021**). Modified Laplace decomposition for time–fractional Schrödinger equations, *Physica Scripta,* **96** 105204.

M. Caputo and F. Mainardi (**2020**). A new fractional model of anomalous diffusion, *Fractional Calculus and Applied Analysis,* **23** 1–15.

M. Dehghan *et al.,* (2021). Fractional Burgers equations: analytical and numerical aspects, *Mathematics and Computers in Simulation,* **179** 198–217.

M. Inc, A. Yusuf, and D. Baleanu (**2022**). Approximate solutions of coupled nonlinear PDEs via Laplace–Adomian," *Mathematics*, **10** 2059.

M. Wazwaz (**2020**). Adomian decomposition and Laplace transform techniques for nonlinear PDEs, *Applied Mathematical Modelling,* **377** 125–135.

N. Kumar and P. Singh (**2021**). Hybrid Sumudu–Laplace transform for viscoelastic fluid models. *Results in Physics,* **26** 104307.

P. Agarwal, T. Abdeljawad, and D. Baleanu (**2023**). Developments in Atangana–Baleanu fractional models, *Mathematics*, **11** 457.

Podlubny (2020). *Fractional Differential Equations*. Academic Press.

R. Herrmann (2021). *Fractional Calculus: An Introduction for Physicists*. World Scientific.

R. Metzler (2022). Fractional dynamics in physics: applications in anomalous transport. *Annals of Physics*, 534, 1–18.

R. Shah *et al.,* (**2022**). VITM for fractional telegraph equations, *Fractional Calculus and Applied Analysis,* **25** 212–229.

S. Momani and Z. Odibat (**2020**). Fractional VITM for nonlinear PDEs, *Chaos Solitons Fractals*, **133** 109–116.

S. Rezapour, H. Aydi, and D. Baleanu (**2022**). Existence and uniqueness analysis of ABCfractional PDEs via fixed point. *Mathematical Methods in the Applied Sciences,* **45** 437–456.

Saadatmandi and M. Dehghan (**2020**). The Laplace decomposition method for fractional PDEs. *Computers & Mathematics with Applications,* **79** 1390–1403.

T. Abdeljawad (**2022**). Applications of VITM to fractional Schrödinger equations, *Mathematics*, **10** 915.

X. Chen (**2021**). Fractional KdV–Burgers equations and applications in shallow water waves. *Ocean Engineering,* **235** 109–124.

Y. Khan (**2020**). Iterative techniques for fractional differential equations: a review, *Mathematical Methods in the Applied Sciences,* **43** 7831–7852.

Z. Wang *et al.,* (2020). Modeling anomalous transport using fractional PDEs, *Nonlinear Dynamics,* 102 801–817.